Lecture 7: Vector Spaces (part 1)
(book: 4.1, 4.2).

Previous lecture: Determinants

Applications of Vector Spaces:
* Vector Space Search Engines. (see the YouTube video on Canvas)
* Digital Signal Processing (Section 4.7)
* Fourier Series (follow-up course Numerical Mathematics)

Intro to Vector Space: 3Blue1Brown - Abstract vector spaces (see Canvas)

So far: $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n, \ldots$ were our vector spaces.

* adding vectors in $\mathbb{R}^n \to$ another vector in $\mathbb{R}^n$.
* scaling vectors in $\mathbb{R}^n \to$ another vector in $\mathbb{R}^n$.

Definition of a vector space V: (axioms)

A non-empty set of objects (vectors) with the following 10 rules:

1. $\underline{u}, \underline{v} \in V \Rightarrow \underline{u}+\underline{v}$ is another vector in V (closed under addition)
2. $\underline{u} \in V, c \in \mathbb{R} \Rightarrow c \cdot \underline{u}$ is another vector in V (closed under scalar mult.)
3. $\underline{u}+\underline{v} = \underline{v}+\underline{u}$ (commutativity)
4. $\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}$ (associativity)

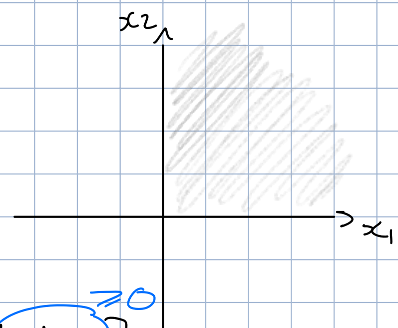
5. + 6. $\exists\, \underline{0} \in V$: $\underline{u}+\underline{0}=\underline{u}$ and $\underline{u}+(-\underline{u})=\underline{0}$ (zero vector)

7. $1 \cdot \underline{u} = \underline{u}$

8. - 10. see the book.

Example: Is $\mathbb{R}^2_+$ a vector space? No.

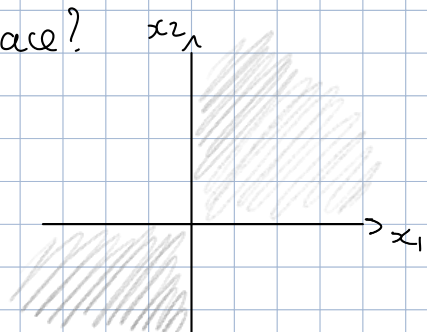
$\hookrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \right\}$



2. ✗ e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2_+$, but $-1 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin \mathbb{R}^2_+$

1. ✓ Let $\underline{u}, \underline{v} \in \mathbb{R}^2_+$. Then, $\underline{u}+\underline{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \overbrace{u_1+v_1}^{\geq 0} \\ \underbrace{u_2+v_2}_{\geq 0} \end{bmatrix}$

Example: Is $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \cdot x_2 \geq 0 \right\}$ a vector space?



1. ✗ e.g. $\underline{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

Then $\underline{u}+\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad 1 \cdot (-1) < 0$.

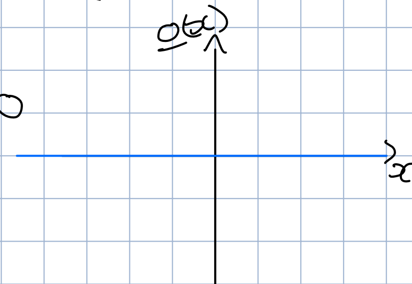
Example: Is $\mathbb{P}_n$ a vector space?
↳ set of polynomials of degree at most n.
i.e., all polynomials of the form
$p(x) = a_0 + a_1 x + a_2 \cdot x^2 + a_3 \cdot x^3 + \cdots + a_n \cdot x^n$.

$n=3$ $\quad \mathbb{P}_3$ $\quad p(x) = x^3 \in \mathbb{P}_3$
$8.5x^4 \notin \mathbb{P}_3$
$9x^2 + x^3 \in \mathbb{P}_3$.
$7.5x^3 + x^4 \notin \mathbb{P}_3$

① Let $p, q \in \mathbb{P}_n$ $\quad p+q \in \mathbb{P}_n$? $\quad p(x) = a_0 + a_1 x + \cdots + a_n \cdot x^n$
$q(x) = b_0 + b_1 x + \cdots + b_n \cdot x^n$.
$(p+q)(x) = p(x) + q(x)$
$= (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n$
So, $p+q \in \mathbb{P}_n$. ✓

② Let $p \in \mathbb{P}_n$ and $c \in \mathbb{R}$. $\quad p(x) = a_0 + a_1 x + \cdots + a_n \cdot x^n$
$(c \cdot p)(x) = c \cdot p(x) = c \cdot (a_0 + a_1 x + \cdots + a_n \cdot x^n)$
$= c \cdot a_0 + (c \cdot a_1) \cdot x + \cdots + (c \cdot a_n) \cdot x^n$.
So, $c \cdot p \in \mathbb{P}_n$ ✓

③-⑩ DIY ✓ $\quad$ But what is $\underline{0}$? $\quad \underline{0}(x) = 0$



Similarly: $\mathbb{P}$ is also a vector space.
↳ set of all polynomials

Example: Set of polynomials of the form
$p(x) = 8 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
① ✗ → So, not a vector space.
② ✗
⑤ ✗

Let V be a vector space and let $W \subseteq V$.
When is W also a vector space?
① $\underline{w}_1, \underline{w}_2 \in W \Rightarrow \underline{w}_1 + \underline{w}_2 \in W$. (closed under addition)
② $\underline{w} \in W$, $c \in \mathbb{R} \Rightarrow c \cdot \underline{w} \in W$ (closed under scalar mult.)
③ $\underline{0} \in W$ (if $W \neq \emptyset$, then ③ follows from ②)
(All the other axioms are fulfilled because $W \subseteq V$ and V is a vector space).
W is called a subspace of V.

Example: Vector space V. $W = \emptyset$ $W \subseteq V$.

Is W a vector space?
Is W a subspace of V?

① ✓
② ✓
③ ✗ (because $W = \emptyset$ and thus $\underline{0} \notin W$) $\rightarrow$ So, W is not a subspace of V.

Example: $\mathbb{R}^3$ is a vector space.
$W = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ $W \subseteq \mathbb{R}^3$.

① ✗
② ✗
③ ✗ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$. $\rightarrow$ So, W is not a subspace of $\mathbb{R}^3$.

Example: $W = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

Example: $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$ Subspace of $\mathbb{R}^3$? Yes.

(W is a line in $\mathbb{R}^3$). ✓

Theorem: Let V be a vector space.
If $\underline{v}_1, \dots, \underline{v}_p \in V$, then $W = \text{Span}\{\underline{v}_1, \dots, \underline{v}_p\}$ is a subspace of V.

Proof:
⓪ We need to show $W \subseteq V$.
Since V is a vector space:
So, $c_1 \underline{v}_1 \in V, \dots, c_p \underline{v}_p \in V$. (because of property ② of V).
So, $c_1 \underline{v}_1 + \dots + c_p \underline{v}_p \in V$ (because of prop ① of V)
So, $W \subseteq V$. ✓

①. If $\underline{x}, \underline{y} \in W$, then $\underline{x} + \underline{y} = c_1 \cdot \underline{v}_1 + \dots + c_p \cdot \underline{v}_p + d_1 \underline{v}_1 + \dots + d_p \underline{v}_p$
$= (c_1 + d_1)\underline{v}_1 + \dots + (c_p + d_p)\underline{v}_p$.
So, $\underline{x} + \underline{y} \in W$ ✓

②. If $\underline{x} \in W$, $a \in \mathbb{R}$, then $a \cdot \underline{x} = a \cdot (c_1 \cdot \underline{v}_1 + \dots + c_p \cdot \underline{v}_p)$
$= (a \cdot c_1) \cdot \underline{v}_1 + \dots + (a \cdot c_p)\underline{v}_p$
So, $a \cdot \underline{x} \in W$. ✓

③. $\underline{0} = 0 \cdot \underline{v}_1 + \dots + 0 \cdot \underline{v}_p$ So, $\underline{0} \in W$. ✓ $\square$

Theorem: Let A be an $m \times n$ matrix.
$\underbrace{\text{The set of all solutions of } A\underline{x}=\underline{0}}_{=\mathrm{Nul}(A).\quad W.}$ is a subspace of $\mathbb{R}^n$.

Proof:

(0). A is $m \times n$, so $\underline{x}$ needs to have n elements. So, $W \subseteq \mathbb{R}^n$. ✓

(1). If $\underline{u}, \underline{v} \in W$, then $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$ ✓

$\underline{u} \in W \Rightarrow A\underline{u} = \underline{0}$

$\underline{v} \in W \Rightarrow A\underline{v} = \underline{0}$

(2). If $\underline{u} \in W$ and $c \in \mathbb{R}$, then $A(c \cdot \underline{u}) = c \cdot A\underline{u} = c \cdot \underline{0} = \underline{0}$ ✓

$\underline{u} \in W \Rightarrow A\underline{u} = \underline{0}$

(3). $A\underline{0} = \underline{0}$. So, $\underline{0} \in W$. ✓

$\square$