Lecture 6 : Determinants
(book: 3.1, 3.2).

Previous lecture: the inverse of a matrix.

Application of the inverse matrix: Cryptography.

Imitation game. A is used to encrypt the message.

Hill algorithm.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ ATTACK-NOW. $\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$ $\begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$

$\begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ $\begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix} \dots$

$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$

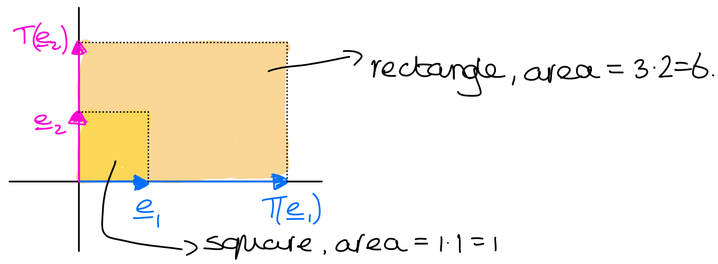
$\begin{bmatrix} A^{-1} \end{bmatrix} \begin{bmatrix} 41 \\ 61 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} A \\ T \end{bmatrix}$.

Inverse of a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

* If $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
  ↳ determinant.
* If $ad - bc = 0$, then A is not invertible
  ↳ singular.

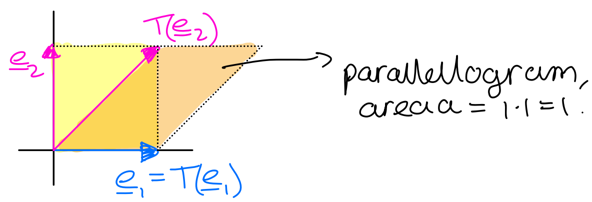
Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\mathbb{R}^2 \to \mathbb{R}^2$

$\uparrow$ $T(e_1)$, $\uparrow$ $T(e_2)$



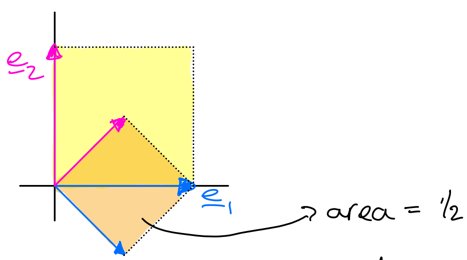
So, A is stretching objects in $\mathbb{R}^2$.
The stretching/scaling factor is $6 = \det(A)$.
↳ >1 because the area increases.

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



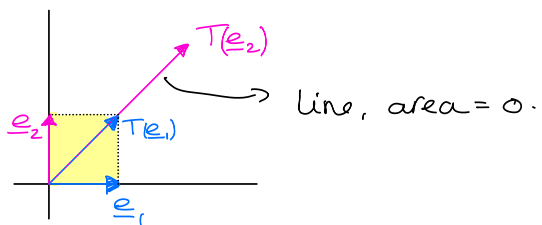
So, $\det(A) = 1$ (because the area stays the same).

Example: $A = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$



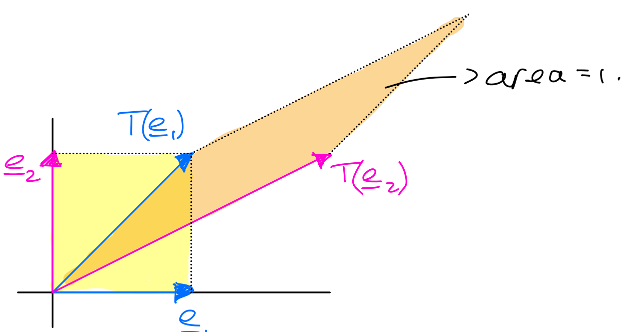
So, $\det(A) = 1/2$ (because the area squishes with a factor $1/2$)

Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$



So, $\det(A) = 0$ (because the unit square is crushed in a line).

Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$



The orientation of space has been "inverted"
So, $\det(A) = -1$.

The determinant of a square $(n \times n)$ matrix is a scalar associated with the matrix.
Notation: $\det(A)$ $|A|$.

It measures how the transformation $T: \underline{x} \to A\underline{x}$ "scales" space:
* in $\mathbb{R}^2$ it measures the change in areas of objects by T.
* in $\mathbb{R}^3$ it measures the change in volumes of objects by T.

$\det(A)=0$ $\Rightarrow$ spaces are "flattened" / we are losing one dimension
$\Rightarrow$ range $\neq$ codomain
$\Rightarrow$ transformation is not surjective (onto).
$\Rightarrow$ A is not invertible.

How to compute the determinant?
* Gaussian elimination.
* cofactor expansion.

Recall $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Cofactor expansion for an $n \times n$ matrix:
* Focus on a specific row i or column j.
* For example, for row i:

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}.$$

* a_{ij}: entry of A at location (i,j)
* C_{ij}: (i,j)-cofactor $= (-1)^{i+j} \cdot \det(A_{ij})$
* A_{ij}: submatrix obtained by removing row i and column j.

Example $A = \begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Cofactor expansion across the first row

$$\det(A) = 3 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 5 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 1 \cdot (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$$
$$= 3 \cdot 2 - 5 \cdot 0 + 1 \cdot 0 = 6.$$

Cofactor expansion across the first column.

$$\det(A) = 3 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 0 + 0 = 3 \cdot 2 = 6.$$

So, be smart: choose a row/column with many 0s.

Triangular matrix: the entries below / above the main diagonal are all 0s.

upper triangular | lower triangular.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

Diagonal matrix: a square matrix whose nondiagonal entries are all 0s.

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

For triangular or diagonal matrices, the determinant equals the product of the entries on the main diagonal.

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44} \cdot \ldots \cdot a_{nn}.$$

REF is upper triangular.
So, maybe we can use Gaussian elimination to compute the determinant?

How do row operations change the determinant?

* two rows of A are interchanged to produce B: $\det(B) = -\det(A)$.
* one row of A is multiplied by k to produce B: $\det(B) = k \cdot \det(A)$
* a multiple of one row of A is added to another row to produce B: $\det(B) = \det(A)$.

Example:

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} \underset{R_1 \leftrightarrow R_3}{=} -1 \cdot \begin{vmatrix} 2 & 4 & 1 \\ 4 & -3 & 0 \\ 0 & 5 & 1 \end{vmatrix} \underset{R_2 : R_2 - 2 \cdot R_1}{=}$$

$$-1 \cdot \begin{vmatrix} 2 & 4 & 1 \\ 0 & -11 & -2 \\ 0 & 5 & 1 \end{vmatrix} \underset{R_3 : R_3 + 5/11 \cdot R_2}{=} -1 \cdot \begin{vmatrix} 2 & 4 & 1 \\ 0 & -11 & -2 \\ 0 & 0 & 1/11 \end{vmatrix} = (-1) \cdot 2 \cdot (-11) \cdot 1/11 = 2.$$

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square matrix A not invertible

$\Rightarrow$ A is not row equivalent to I_n.

$\Rightarrow$ a pivot is missing.

$\Rightarrow$ $\det(\text{REF of } A) = 0$.

$\Rightarrow$ $\det(A) = (-1)^{\#\text{swaps}}$ $\quad \det(\text{REF of } A) = (-1)^{\#\text{swaps}} \cdot 0 = 0$

$\Rightarrow$ $\det(A) = 0$.

Conclusion: square matrix A is not invertible $\Leftrightarrow$ $\det(A) = 0$.

Properties of determinants:

* $\det(A^T) = \det(A)$
* $\det(AB) = \det(A) \cdot \det(B)$ (Thm 6).
* but $\det(A+B) \neq \det(A) + \det(B)$ in general.
* $\det(c \cdot A) = c^n \cdot \det(A)$

Theorem: $\det(A^{-1}) = \dfrac{1}{\det(A)}$ (for all invertible matrices).

Proof:

$$I_n = A \cdot A^{-1}$$

$$\Rightarrow \det(I_n) = \det(A \cdot A^{-1})$$

$$\Rightarrow 1 = \det(A) \cdot \det(A^{-1})$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A^{-1}) \qquad \square$$

Summary (so far):

Let A be an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$.

$m \geq n$: $\begin{bmatrix} A \end{bmatrix}$

The following statements are equivalent:

1. A has a pivot in every column
2. A has n pivot positions
3. There are no free variables
4. $A\underline{x} = \underline{0}$ has only the trivial sol.
5. $\{\underline{a}_1, \underline{a}_2, \cdots \underline{a}_n\}$ is linearly indep.
6. $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/injective

$m \leq n$: $\begin{bmatrix} A \end{bmatrix}$

The following statements are equivalent:

a. A has a pivot in every row.
b. A has m pivot positions.
c. The echelon form of A does not contain a row of all zeros.
d. $A\underline{x} = \underline{b}$ is consistent for every $\underline{b}$ in $\mathbb{R}^m$.
e. $\text{Span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} = \mathbb{R}^m$.
f. $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.

The Invertible Matrix Theorem:

If A is square ($n = m$), then statements 2 and b are equivalent.
Hence, the following statements are equivalent for square matrices

* 1 - 6, a - f
* A is invertible
* There is a matrix C such that $CA = I_n$ and $AC = I_n$
* A is row equivalent to I_n.
* A^T is invertible.
* $\det A \neq 0$