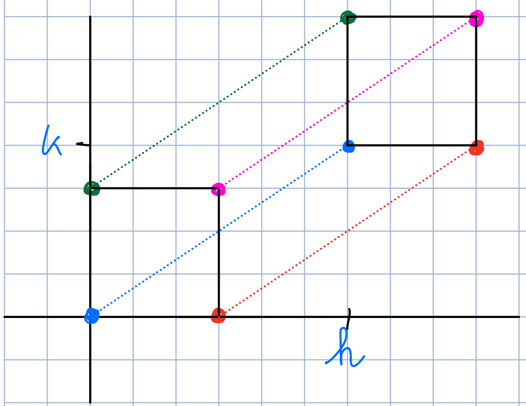
Lecture 5: Perspective projections + The Inverse of a Matrix
(book: 2.2, 2.3, 2.7).

Previous lecture: Linear transformations, Matrix algebra

Applications to Computer Graphics.



This is a transformation, but not linear because $T(\underline{0}) \neq \underline{0}$.

We cannot find a 2x2 matrix A such that $A\underline{x} = T(\underline{x}) = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$

NOT possible

$$\begin{bmatrix} 1 & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$$

Homogeneous coordinates: work on a specific plane in $\mathbb{R}^3$. For example the plane with $x_3 = 1$.

(Note: a plane in $\mathbb{R}^3$ is not $\mathbb{R}^2$).

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \\ 1 \end{bmatrix}$$

LAB 2: Perspective projections
↳ project a 3-dimensional object on a 2-dimensional computer screen.

(not on the exam).

Recall the three row operations,
* replacement
* scaling
* interchange.

$A \sim A_2 \sim A_3 \sim A_4 \sim A_5$.

Can we find an elementary matrix E such that $EA = A_5$.

Example: $\underset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}} \underset{1}{\sim} \underset{A_2}{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}} \underset{2}{\sim} \underset{A_3}{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}} \underset{3}{\sim} \underset{A_4}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \underset{4}{\sim} \underset{A_5}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

1. $R_1 \leftrightarrow R_2$
2. $R_2 : R_2 - 2 \cdot R_1$
3. $R_2 : R_2 * -1$
4. $R_1 : R_1 - 2 \cdot R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{1}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1 \qquad E_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{2}{\sim} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_2 \qquad E_2 A_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = A_3$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_3 \qquad E_3 A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = A_4$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{4}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = E_4 \qquad E_4 A_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_5$

Hence, $A_5 = E_4 A_4 = E_4 (E_3 A_3) = E_4 E_3 (E_2 A_2) = E_4 E_3 E_2 (E_1 A)$
$= \underbrace{E_4 E_3 E_2 E_1}_{E} A = EA$.

Check: $E = \ldots = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \qquad EA = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So, E takes us to the reduced echelon form of A.

So, E is obtained by applying the same row operations on the identity matrix.

An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix C such that $CA = I_n$ and $AC = I_n$.

This C is the inverse of A.
The inverse of a matrix is unique.
Notation: A^{-1}

For a matrix A to be invertible:
* A must be square ($n\times n$)
* the RREF of A is the identity matrix: $A \sim \dots \sim I_n$ (Thm 7)
  (A has a pivot position in every row/column: A has n pivot positions).

How to find the inverse of an invertible matrix A?
We need to find E such that $EA = I_n$.

$$\begin{array}{c} \\ n \end{array}\overset{2n}{\left[A \,\vdots\, I_n\right]} \sim E\left[A \,\vdots\, I_n\right] = \left[EA \,\vdots\, EI_n\right] = \left[I_n \,\vdots\, E\right]$$ hey there is A^{-1} :)

An algorithm for finding A^{-1}:
* Row reduce $\left[A \,\vdots\, I_n\right]$
* If this leads to $\left[I_n \,\vdots\, E\right]$, then A is invertible and $A^{-1} = E$.
* Otherwise, A is not invertible.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ inverse of A?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underset{R_2 : R_2 - 2\cdot R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

A is not invertible because we cannot reduce it to I_2.

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ inverse of A?

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \underset{R_1 : R_1 - 2R_2}{\sim} \left[\begin{array}{cc:cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

$AA^{-1} = I_n$ ✓

Thm: The inverse of a matrix is unique.
Proof: Let A be an $n\times n$ matrix.
Suppose B and C are both an inverse of A and $B \neq C$.
$B = BI_n = B(AC) = (BA)C = I_nC = C$ ↯ □

Thm: Let A be an $n\times n$ invertible matrix. Then, for each $\underline{b} \in \mathbb{R}^n$, the equation $A\underline{x} = \underline{b}$ has the unique solution $\underline{x} = A^{-1}\underline{b}$.

Proof:
* existence: A is invertible $\Rightarrow$ A has a pivot in every row $\Rightarrow$
  $A\underline{x} = \underline{b}$ is consistent for every $\underline{b} \in \mathbb{R}^n$.
* uniqueness: A is invertible $\Rightarrow$ A has a pivot in every col. $\Rightarrow$
  no free variables $\Rightarrow$ solution must be unique.
* Show that $\underline{x} = A^{-1}\underline{b}$ is the unique solution:

$A\underline{x} = \underline{b}$
$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$

$A\underline{x} = A(A^{-1}\underline{b}) = (AA^{-1})\underline{b} = I_n\underline{b} = \underline{b}$ ✓

$$(A^{-1}A)\underline{x} = A^{-1}\underline{b}$$
$$I_n \underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1}\underline{b} \ ✓$$

$\square$

Thm: Let A be an $n\times n$ matrix.
If $CA = I_n$, then also $AC = I_n$.

Proof: Assume $CA = I_n$.

Claim / Lemma: For each $\underline{b} \in \mathbb{R}^n$, $A\underline{x} = \underline{b}$ has a solution.
Proof: Suppose $\exists\, \underline{b} \in \mathbb{R}^n$ st $A\underline{x} = \underline{b}$ has no solution.
Then, A does not have a pivot in every row.
So, A also does not have a pivot in every column.
So, there is a nontrivial solution $\underline{y}$: $A\underline{y} = \underline{0}$ with $\underline{y} \neq \underline{0}$.
Then, $\underline{y} = I_n \underline{y} = (CA)\underline{y} = C(A\underline{y}) = C\underline{0} = \underline{0}$ ↯
So, $A\underline{x} = \underline{b}$ has a solution for every $\underline{b} \in \mathbb{R}^n$ $\square$

Let $\underline{b} \in \mathbb{R}^n$. $A\underline{x} = \underline{b}$ has a sol.
$\Rightarrow C(A\underline{x}) = C\underline{b}$
$\Rightarrow (CA)\underline{x} = C\underline{b}$
$\Rightarrow I_n \underline{x} = C\underline{b}$
$\Rightarrow \underline{x} = C\underline{b}$ So, this is the sol of $A\underline{x} = \underline{b}$.

Hence, $\underline{b} = A\underline{x} = A(C\underline{b}) = (AC)\underline{b}$ So, we need $AC = I_n$ $\square$

Properties of invertible matrices:

* $(A^{-1})^{-1} = A$
* If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$ (verify yourself)
* $(A^T)^{-1} = (A^{-1})^T$.

Summary (so far):

Let A be an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$.

$m \geq n$: $\begin{bmatrix} A \\ \\ \\ \end{bmatrix}$

The following statements are equivalent:

1. A has a pivot in every column
2. A has n pivot positions
3. There are no free variables
4. $A\underline{x} = \underline{0}$ has only the trivial sol.
5. $\{\underline{a}_1, \underline{a}_2, \ldots \underline{a}_n\}$ is linearly indep.
6. $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/injective

$m \leq n$: $\begin{bmatrix} & & A & & \end{bmatrix}$

The following statements are equivalent:

a) A has a pivot in every row.
b) A has m pivot positions.
c) The echelon form of A does not contain a row of all zeros.
d) $A\underline{x} = \underline{b}$ is consistent for every $\underline{b}$ in $\mathbb{R}^m$.
e) $\text{Span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} = \mathbb{R}^m$.
f) $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.

The Invertible Matrix Theorem:

If A is square ($n=m$), then statements 2 and b are equivalent. Hence, the following statements are equivalent for square matrices

* 1 – 6, a – f
* A is invertible
* There is a matrix C such that $CA = I_n$ <u>and</u> $AC = I_n$
* A is row equivalent to I_n.
* A^T is invertible.
* $\det A \neq 0$