Lecture 2: Vector equations, Matrix equations.
(book: 1.3, 1.4)

Previous lecture: geometric / row point of view to an SLE
+ Gaussian elimination.

Today: column point of view to an SLE.

$$\begin{aligned} x + y &= 30 \\ 2x + 4y &= 74 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 30 \\ 74 \end{bmatrix}$$

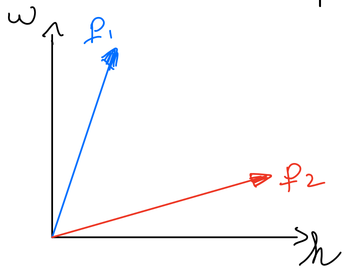
From a machine learning perspective: columns of A are feature vectors (vectors that collect features that we measured for each animal).

For instance, height and weight $\underline{f} = \begin{bmatrix} h \\ w \end{bmatrix}$

Collect data from N animals $\rightarrow$ N feature vectors:

$$\underline{f}_1 = \begin{bmatrix} h_1 \\ w_1 \end{bmatrix}, \ \underline{f}_2 = \begin{bmatrix} h_2 \\ w_2 \end{bmatrix}, \ \ldots\ldots, \ \underline{p}_N = \begin{bmatrix} h_N \\ w_N \end{bmatrix}.$$

These vectors are points in the feature space $(\mathbb{R}^2)$.

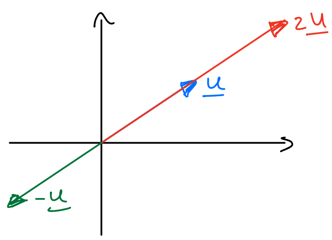


In general: n features $\rightarrow$ $(n \times 1)$ vector $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$

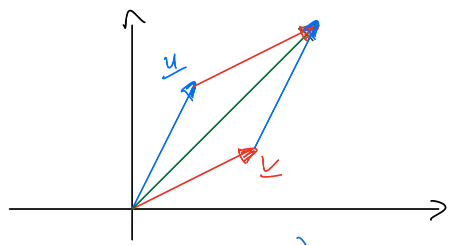
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can perform operations with vectors:

* scaling $\quad \underline{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad 2\underline{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \qquad -\underline{u} = (-1)\cdot\underline{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$



* addition $\underline{u} = \begin{bmatrix}1\\2\end{bmatrix}$ $\underline{v} = \begin{bmatrix}2\\1\end{bmatrix}$ $\underline{u}+\underline{v} = \begin{bmatrix}3\\3\end{bmatrix}$



(parallellogram rule)

Algebraic properties of $\mathbb{R}^n$: book p53.

Combining these two operations:

Given vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ in $\mathbb{R}^n$ and $c_1, \dots, c_p$ the vector

$$\underline{y} = c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_p \cdot \underline{v}_p$$

is called a linear combination of $\underline{v}_1, \dots, \underline{v}_p$ with weights $c_1, \dots c_p$.

Example: $\underline{u} = \begin{bmatrix}1\\2\end{bmatrix}$ $\underline{v} = \begin{bmatrix}2\\1\end{bmatrix}$ $\underline{b} = \begin{bmatrix}5\\4\end{bmatrix}$

Q: can $\underline{b}$ be written as a linear combination of $\underline{u}$ and $\underline{v}$?
i.e., can we find c_1 and c_2 such that $c_1 \cdot \underline{u} + c_2 \cdot \underline{v} = \underline{b}$?

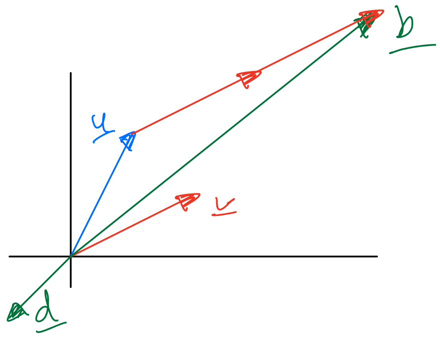
$$c_1 \cdot \begin{bmatrix}1\\2\end{bmatrix} + c_2 \cdot \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix} \text{ ?} \quad \leftarrow \text{vector equation}$$

$$\begin{bmatrix}c_1\\2c_1\end{bmatrix} + \begin{bmatrix}2c_2\\c_2\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}$$

$$\begin{bmatrix}c_1 + 2c_2\\2c_1 + c_2\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = 5 \\ 2c_1 + c_2 = 4 \end{cases} \quad \leftarrow \text{hey. that is an SLE !}$$

* Every SLE can be written as a vector equation, and the other way around.
* Solving an SLE means investigating whether $\underline{b}$ can be written as a linear combination of the columns of $\underline{A}$.
(column point of view).



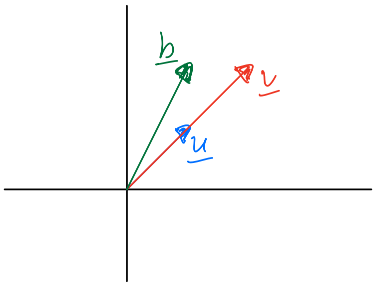
$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\underline{b} = \underline{u} + 2 \cdot \underline{v}.$

$\underline{d} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$-1/3 \ \underline{u} + (-1/3) \ \underline{v}$

Example: (no sol.) $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\underline{b} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



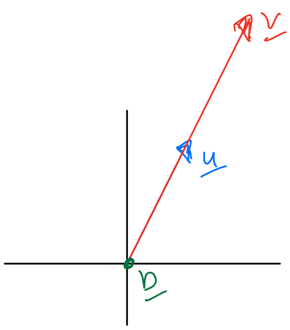
Hence, there is no way to obtain $\underline{b}$ by taking a linear combination of $\underline{u}$ and $\underline{v}$.

Example (∞ many solutions in $\mathbb{R}^2$): $\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2 \cdot \underline{u} + (-1) \cdot \underline{v} = \underline{b}$

$0 \cdot \underline{u} + 0 \cdot \underline{v} = \underline{b}$

$u + (-1/2) \cdot \underline{v} = \underline{b}.$



There are ∞ many ways to linearly combine $\underline{u}$ and $\underline{v}$.

$$\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \underset{R_2 : R_2 - 2 \cdot R_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

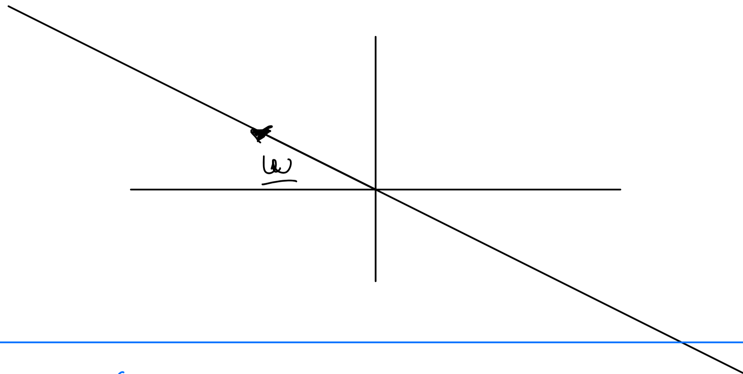
$$\begin{cases} x_1 = -2x_2 \\ x_2 \text{ is free} \end{cases}$$ (parametric form)

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$ (parametric vector form)

Hence, the solution set is $x_2 \cdot \underline{w}$, where $\underline{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

i.e., Span $\{\underline{w}\}$

↳ any scalar multiple of $\underline{w}$.



any solution on this line is a solution to the SLE.

Example (∞ many solutions in $\mathbb{R}^3$)

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 - 6x_2 + 4x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 0 \end{array}\right] \sim_{R_2 : R_2 - 2\cdot R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 3x_2 - 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$ parametric form

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{\underline{w}_1} + x_3 \cdot \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{w}_2}$$

Hence, a solution is any linear combination of $\underline{w}_1$ and $\underline{w}_2$.

Span $\{\underline{w}_1, \underline{w}_2\}$

So, the solution set is a plane in $\mathbb{R}^3$.

Given a set of p vectors $\{\underline{v}_1, \dots, \underline{v}_p\}$ in $\mathbb{R}^n$, the span of this set of vectors is the set of all possible linear combinations of the vectors in this set.

⟶ Span $\{\underline{v}_1, \dots, \underline{v}_p\}$ contains any vector $\underline{y}$ that can be written as

$$\underline{y} = c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_p \cdot \underline{v}_p.$$

Hence, solving an SLE boils down to investigating whether $\underline{b}$ belongs to the Span of the columns of A.

We can see a linear combination of vectors as the product of a matrix (A) and a vector ($\underline{x}$).

Definition of $A\underline{x}$: linear combination of the columns of A with the entries of $\underline{x}$ being the weights.

$$A\underline{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

More efficient $A\underline{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

We need: # columns of A = # row/entries of $\underline{x}$.

Properties of the matrix-vector product: p. 65

Three things with the same solution set:

* the SLE with augmented matrix $[\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n \ \vdots \ \underline{b}]$
* the vector equation $x_1 \cdot \underline{a}_1 + x_2 \cdot \underline{a}_2 + \cdots + x_n \cdot \underline{a}_n = \underline{b}$.
* the matrix equation $[\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{b}$. $A\underline{x} = \underline{b}$

Example: $A = \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -2 \\ -2 & -10 & 6 \end{bmatrix}$

Q: Is $A\underline{x} = \underline{b}$ consistent for every $\underline{b} \in \mathbb{R}^3$? No!

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & b_1 \\ 0 & 1 & -2 & b_2 \\ -2 & -10 & 6 & b_3 \end{array}\right] \underset{R_3:\ R_3 + 2 \cdot R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 5 & -3 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 + 2b_1 \end{array}\right]$$

The SLE is consistent iff $b_3 + 2b_1 = 0$.

For example, the SLE is inconsistent if $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

consistent if $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

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4 equivalent statements:

* The columns of A span $\mathbb{R}^m$.
* Each $\underline{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
* For each $\underline{b} \in \mathbb{R}^m$, $A\underline{x} = \underline{b}$ has a solution.
* A has a pivot position in every row.
  ↳ (coefficient matrix!)