Lecture 1: Systems of linear equations, Gaussian elimination.
(book: 1.1, 1.2)

x: chickens.
y: cows.

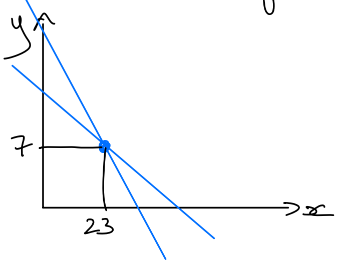
$$\begin{cases} x+y=30 \\ 2x+4y=74 \end{cases}$$

This is a system of linear equations (SLE)

2 equations in 2 variables.
We need a solution (a pair (x,y)) that works for both equations.

We have 2 variables, so we "live" in $\mathbb{R}^2$.
Each equation of this SLE represents a line in $\mathbb{R}^2$.

From a geometric/row point of view: we're looking for an intersection of those two lines.



In general solving an SLE means:
* $\mathbb{R}^2$: find intersection of lines in a plane.
* $\mathbb{R}^3$: " " " planes in a 3d-space.
* $\mathbb{R}^n$: " " " hyperplanes in a hyperspace

} a geometric/row point of view.

$$\begin{cases} x+y=30 \\ 2x+4y=74 \end{cases} \longrightarrow \begin{cases} x+y=30 \\ 2x+4y-2\cdot(x+y)=74-2\cdot 30 \end{cases} \longrightarrow \begin{cases} x+y=30 \\ \tfrac{1}{2}*2y=14*\tfrac{1}{2}. \end{cases}$$

$$\begin{cases} x+y-y=30-7 \\ y=7 \end{cases} \longrightarrow \begin{cases} x = 23 \\ y=7 \end{cases}$$

So, the SLE has a solution and solution is unique.

Efficient procedure to solve an SLE: Gaussian elimination/row reduction.

An SLE can be summarized by:
* coefficient matrix A
* vector $\underline{b}$ of RHS numbers.

} augmented matrix $[A \,\vdots\, \underline{b}]$

$$\begin{cases} x+y=30 \\ 2x+4y=74 \end{cases} \qquad A=\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \qquad \underline{b}=\begin{bmatrix} 30 \\ 74 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & 30 \\ 2 & 4 & 74 \end{array}\right]$$

2 × 2 matrix
\# rows × \# columns.
\# equations × \# variables.
$m \times n$

$$[A \vdots \underline{b}] = \begin{bmatrix} 1 & 1 & \vdots & 30 \\ 2 & 4 & \vdots & 74 \end{bmatrix} \underset{R_2 : R_2 - 2 \cdot R_1}{\sim} \begin{bmatrix} 1 & 1 & \vdots & 30 \\ 0 & 2 & \vdots & 14 \end{bmatrix} \underset{R_2 : R_2 * 1/2}{\sim} \begin{bmatrix} 1 & 1 & \vdots & 30 \\ 0 & 1 & \vdots & 7 \end{bmatrix}$$

$$\underset{R_1 : R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & \vdots & 23 \\ 0 & 1 & \vdots & 7 \end{bmatrix} \qquad \begin{cases} 1 \cdot x + 0 \cdot y = 23 \\ 0 \cdot x + 1 \cdot y = 7 \end{cases} \qquad \begin{cases} x = 23 \\ y = 7 \end{cases}$$

So, the unique solution is $\begin{cases} x = 23 \\ y = 7 \end{cases}$

The corresponding SLEs of these augmented matrices are all equivalent ($\sim$) to each other: they share the same solution set.

Two matrices are row equivalent if one matrix can be changed into the other by means of a row operation:

① Replacement: add a scalar multiple of a row to another row.

② Scaling: multiply a row by a <u>nonzero</u> scalar.

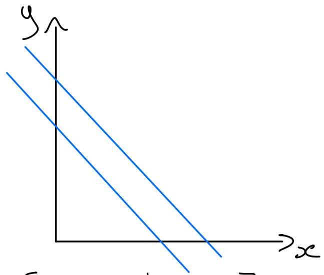
③ Interchange: swap two rows.

If the augmented matrices of two SLEs are row equivalent, then the two SLEs are equivalent.

An SLE can have:

* one unique solution } SLE is consistent
* infinitely many solutions }
* no solution } SLE is inconsistent.

$$\begin{cases} x+y=30 \\ x+y=49 \end{cases}$$

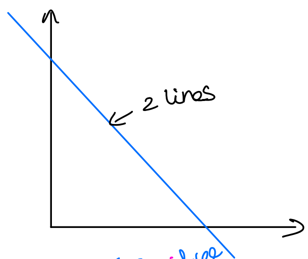


two parallel lines
→ no intersection
→ no solution
→ SLE is inconsistent.

$$\left[\begin{array}{cc:c} 1 & 1 & 30 \\ 1 & 1 & 49 \end{array}\right] \underset{R_2 : R_2 - R_1}{\sim} \left[\begin{array}{cc:c} 1 & 1 & 30 \\ 0 & 0 & 19 \end{array}\right] \qquad 0 = 19 \ \lightning$$

such a row $[0\,0 \cdots 0 \,\vdots\, \alpha]$, $\alpha \neq 0$, is typical for an inconsistent SLE.

$$\begin{cases} x+y=30 \\ 2x+2y=60 \end{cases}$$



they are the same line
→ infinitely many points of intersection
→ infinitely many solutions.

$$\left[\begin{array}{cc:c} 1 & 1 & 30 \\ 2 & 2 & 60 \end{array}\right] \underset{R_2 : R_2 - 2R_1}{\sim} \begin{array}{cc} \text{basic var} & \text{free var.} \\ \left[\begin{array}{cc:c} 1 & 1 & 30 \\ 0 & 0 & 0 \end{array}\right] & \end{array} \qquad \begin{cases} x+y=30 \\ 0=0 \end{cases} \ ☺$$

Solution set in parametric form $\begin{cases} x = 30 - y. \\ y \text{ is free} \end{cases}$

y is a free variable (we can choose any value we want)
x is a basic variable (we cannot choose it)

Note: having a zero does not necessarily mean that there are infinitely many solutions.
For example:

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x=1 \\ y=0 \end{cases}$$

So, the SLE has a unique sol.

Let's formalize the row reduction / Gaussian elimination algorithm:

$\boxed{\text{pivot (leading entry): leftmost nonzero element in a row}}$

Row echelon form (REF):

(not unique for a matrix)

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

one unique sol.

no solution (because there is a pivot in the last column).

① All nonzero rows are above any zero row.
② Every pivot in a row is in a column to the right of the pivot of the row above it.
③ All entries below a pivot are zero.

It tells you:

* whether there is a solution (existence question)
  → is there a row $[0 \cdots 0 \,|\, \alpha]$ with $\alpha \neq 0$? (last column is a pivot column)

* whether the solution is unique (uniqueness question)
  → are there free variables?
  → are all variables basic variables?
  → does every column in the coefficient matrix have a pivot?

Reduced row echelon form (RREF):

(unique for a matrix).

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{cases} x_1 = 2 \\ x_2 = 0 \\ x_3 = -1 \end{cases} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 = -1 - x_3 \\ x_2 = -2 \\ x_3 \text{ is free} \end{cases}$$

④ All pivots are equal to 1.
⑤ Each pivot is the only nonzero entry in its column.

It tells you:

* the solution.

Example (Gaussian elimination algorithm):

$$\begin{cases} \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ x_1 - 2x_2 + x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \underset{R_3 : R_3 + 4 \cdot R_1}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \underset{R_3 : R_3 + 3/2\, R_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \begin{array}{l} R_1 : R_1 - R_3 \\ R_2 : R_2 + 8R_3 \end{array}$$

echelon form!
the SLE is consistent
the solution is unique.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \quad R_2 : R_2 * 1/2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \quad R_1 : R_1 + 2 \cdot R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

reduced echelon form!

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}.$$