Numbers:

Integer: A whole number or a number that can be written without a decimal expression or a quotient, e.g. $1, 7, -3, 28, 0$

The set of integers is denoted by $\mathbb{Z}$. In set notation (which will be used more often from chapter 2 on): $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. The braces $\{\}$ denote a set; the items in the set are called elements and they are separated by commas. The sign $\in$ means "is an element of", so $-37 \in \mathbb{Z}$.

Rational Number: A number that can be written as the quotient of 2 integers, e.g. $\frac{3}{7}, \frac{-4}{29}, 3 = \frac{3}{1}$.

The set of rational numbers is denoted by $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\}$. Here the colon : means "for which" or "such that" (in this case such that).

Natural Number: A positive integer, so any integer $n \geq 1$.

Note: Some books also include 0 in the set of natural numbers, but we do NOT.

The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Even Number: An integer that is divisible by 2.

Any even number m is equal to $2 \cdot n$, or $m = 2n$, for an appropriate integer n.

Odd Number: An integer m that is not an even number: $m = 2n + 1$ (or $m = 2n - 1$) for an appropriate integer n.

Divisibility: An integer m is divisible by 3, or m is a multiple of 3, if $m = 3n$ for some integer n. Generally: m is divisible by k if $m = k \cdot n$ for some integer n.

Prime (Number): A natural number bigger than 1 that is only divisible by 1 and by itself, e.g. $2, 3, 5, 7, 11$.

(Perfect) Square: A square is an number that is the product of an integer with itself, e.g. $1 = 1 \cdot 1$, $121 = 11 \cdot 11$, $0 = 0 \cdot 0$.

Real Number: A number that can be described by an infinite sequence of decimals (all rational numbers but also $\sqrt{2}$, π etc.

Examples: $\frac{1}{7} = 0.142857142857$ etc, $\pi = 3.141592$ etc, $1 = 1.00000$ etc.

The set of real numbers is denoted by $\mathbb{R}$. This set can be identified with a line (for instance the x-axis). The points on this line form the set $\mathbb{R}$.

Section 1.1

A **proposition** is a formal statement that is either true or false, but that is NOT a matter of opinion.

Example 1 *Examples of propositions in mathematics:*

- $1 + 1 = 2$. *This is a true proposition as* $1 + 1$ *indeed equals* 2.
- $x = 1$ *solves the equation* $x^2 + 4x - 5 = 0$. *This is also a true proposition: If we plug in* $x = 1$ *in the equation* $x^2 + 4x - 5 = 0$*, then it says* $1^2 + 4 \cdot 1 - 5 = 0$*, which is indeed true.*
- $x = 2$ *solves the equation* $x^2 + 4x - 5 = 0$*: FALSE: If we plug in* $x = 2$ *in the equation* $x^2 + 4x - 5 = 0$*, then it says* $2^2 + 4 \cdot 2 - 5 = 0$*, which is false.*
- $x = 1$ *is the unique solution of the equation* $x^2 + 4x - 5 = 0$. *This proposition is false, as* $x = -5$ *also solves it.*
- $2 + 2 = 5$*: FALSE (everyone understands why?)*
- $\sqrt{2}$ *is a rational number: FALSE: We will prove the 'falseness' of this proposition later.*
- *If* x *is a real number, then* x *is positive, negative or zero. TRUE: Indeed, if* x *is a real number, it must be positive, negative or zero; there are no other possibilities.*

Example 2 *The following statements are NOT propositions. For all statements this is the case because we are missing some information that is required to make sure that we can state that it is true or false.*

- $x = 2$.
  This could be true or false, depending on the value of x *(and is* x *even a number?).*
- $x^2 + 4x - 5 = 0$.
  This is true if $x = 1$ *or* $x = -5$*, otherwise it is false.*
- x *is positive, negative or zero.*
  x *might be a tree...*

The exercise below uses the concept of a proposition being true or false in a funny way.

Exercise 1 *Below are 10 propositions, each of which is either true or false. Some of the propositions concern an unknown number* N. *Your task is to find the smallest possible value of* N. *Hint: You should first figure out which propositions are true and which are false and then, based on the set of true propositions, determine the value of* N.

1. *At least one of propositions 9 and 10 is true.*
2. *This is either the first true proposition or the first false proposition.*
3. *There are three consecutive false propositions.*

4. *The difference between the number of the last true proposition and the last false proposition divides the number N (or, equivalently, N is a multiple of said difference).*
5. *The sum of the numbers of the true propositions is equal to the number N.*
6. *This is not the last true proposition.*
7. *The number of each true proposition divides the number N.*
8. *The number N is the percentage of true propositions.*
9. *The number of divisors of N (apart from 1 and N itself) is greater than the sum of the numbers of the true propositions. A divisor of N is a number that divides N.*
10. *There are no three consecutive true propositions.*

A proposition consisting of only a single statement is usually called a primitive proposition. It is also possible to create propositions that consist of several statements. Such propositions are called **compound propositions.** There are three symbols in use to create such propositions:

Symbol	Meaning
not, $\urcorner$, $\sim$	negation
and, $\wedge$, &	conjunction
or, $\vee$	disjunction

Example 3 *Examples of compound propositions:*

- $1 + 1 = 2 \;\vee\; 2 + 2 = 5$ *(TRUE);*
- $(1 + 1 = 2 \;\vee\; 2 + 2 = 5) \;\wedge\; 3 + 3 = 7$ *(FALSE).*

Make sure that you use the braces correctly in compound propositions. The proposition $(1+1 = 2 \;\vee\; 2+2 = 5) \;\wedge\; 3+3 = 7$ is entirely different from the proposition $1+1 = 2 \;\vee\; (2+2 = 5 \;\wedge\; 3+3 = 7)$!

Section 1.2

An unspecified proposition is usually denoted by a letter, like p, q, r etc. For propositions p and q we can create the compound proposition $p \wedge q$ which is only true if p and q are both true. Furthermore the proposition $p \vee \urcorner q$ is true if p is true or $\urcorner q$ is true, so if p is true or q is false.

Example 4 *The two propositions $(p \wedge q) \vee \urcorner r$ and $p \wedge (q \vee \urcorner r)$ are different as will be shown below.*

A **truth table** is a table that shows the truth-value for compound propositions given all possible truth-values for the simple propositions (p, q, r). For example 2 we have:

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$	$q \vee \neg r$	$p \wedge (q \vee \neg r)$
T	T	T	T	F	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	F	F
T	F	F	F	T	T	T	T
F	T	T	F	F	F	T	F
F	T	F	F	T	T	T	F
F	F	T	F	F	F	F	F
F	F	F	F	T	T	T	F

Notice that if there are n simple propositions that can each be true or false a truth table consists of $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2 = 2^n$ rows!

Section 1.3

Implication Arrows: $\Rightarrow$, $\Leftarrow$, $\Leftrightarrow$.

$p \Rightarrow q$: p implies q, if p then q, p only if q. (sufficiency: for q to be true it is sufficient that p is true)

$p \Leftarrow q$: p is implied by q, p if q. (necessity: for q to be true it is necessary that p is true)

$p \Leftrightarrow q$: p if and only if q, p is equivalent to q. True if p and q have the same truth value.

Propositions that make use of implication arrows are called **conditional propositions**.

The following 2 propositions are equivalent:

$$p \Rightarrow q \quad \text{and} \quad \neg q \Rightarrow \neg p$$

This follows immediately from the truth tables of the 2 propositions:

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Exercise 2 *Show that the proposition* $\neg p \vee q$ *is also equivalent to the 2 propositions mentioned above.*

It is possible to construct slightly more complicated propositions, as the following example shows.

Example 5 *Give the truth table for the following proposition:*

$$S = [(p \wedge ((q \vee \neg r) \Rightarrow \neg q)) \Rightarrow (\neg p \vee r)] .$$

p	q	r	$q \vee \neg r$	$\neg q$	$(q \vee \neg r) \Rightarrow \neg q$	$p \wedge ((q \vee \neg r) \Rightarrow \neg q)$	$\neg p \vee r$	S
T	T	T	T	F	F	F	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	T	T	T	T	F	F
F	T	T	T	F	F	F	T	T
F	T	F	T	F	F	F	T	T
F	F	T	F	T	T	F	T	T
F	F	F	T	T	T	F	T	T

Exercise 3 *(exercise 1.3.1 in the book) Give the truth table for the following propositions:*

- $p \Leftrightarrow (q \vee p)$
- $p \Leftrightarrow (q \Rightarrow p)$
- $p \Leftrightarrow (q \Rightarrow r)$
- $S = [((\neg(p \wedge \neg q) \wedge r) \Rightarrow p) \Leftrightarrow (q \wedge \neg r)]$

Exercise 4 *Is the following proposition true or false?*

$$1 + 1 = 3 \;\Rightarrow\; 2 + 2 = 5$$

Exercise 5 *You have been selected to serve on jury for a criminal case. The attorney for the defence argues as follows: "If my client is guilty, then the knife was in the drawer. Either the knife was not in the drawer or Jason Pritchard saw the knife. If the knife was not there on 10 October, it follows that Jason Pritchard did not see the knife. Furthermore, if the knife was there on 10 October, then the knife was in the drawer and also the hammer was in the barn. But we all know that the hammer was not in the barn. Therefore, ladies and gentlemen of the jury, my client is innocent."*

Question: Is the attorney's argument sound? How should you vote?

Section 1.4

There are 2 different **quantifiers:**

- The universal quantifier: $\forall$ (for all)
- The existential quantifier: $\exists$ (there exists)

These quantifiers can be used to construct propositions concerning more than one object at the same time.

Example 6

$$
\begin{aligned}
1: &\quad \exists x \in \mathbb{R} : x^2 \leq \frac{1}{\pi} \\
2: &\quad \forall x \in \mathbb{R} : x^2 \leq \frac{1}{\pi} \\
3: &\quad \forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : 3x + y \leq 4 \\
4: &\quad \exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : 3x + y \leq 4
\end{aligned}
$$

The colon : most of the time should be read as one of the following three phrases: "such that", "for which" or "we have (that)" (or "it holds that"). Now that we know this, it is not difficult anymore to figure out how the above propositions should be "pronounced":

1 There exists a real number x such that $x^2 \leq \frac{1}{\pi}$. This proposition is obviously true.

2 For all real numbers x we have: $x^2 \leq \frac{1}{\pi}$. This proposition is obviously false.
Of course saying that something is obviously true or false is not a proof, but when section 1.5 is discussed propositions like these should cause absolutely no problem anymore!

3 For all integers x there exists an integer y such that $3x+y \leq 4$. This is a lot less obvious. In section 1.5 we will prove that this proposition is true.

4 There exists an integer x such that for all integers y we have: $3x + y \leq 4$. In section 1.5 we will prove that this proposition is false.

Of course it is possible to build bigger propositions using more than 2 variables and quantifiers.

Each of the 4 propositions in the example 3 can be negated. This turns out to be a surprisingly simple procedure. We will negate propositions 1 and 3, 2 and 4 are left as an exercise. The negation of proposition 1:

$$
\begin{aligned}
&\neg \left(\exists x \in \mathbb{R} : x^2 \leq \frac{1}{\pi} \right) \\
&\Leftrightarrow \forall x \in \mathbb{R} : \neg \left(x^2 \leq \frac{1}{\pi} \right)
\end{aligned}
$$

(If there does not exists an x with a particular property then apparently for all x we must have "not this property")

$$\Leftrightarrow \forall x \in \mathbb{R} : x^2 > \frac{1}{\pi}$$

The negation of proposition 3:

$$
\begin{aligned}
&\neg (\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : 3x + y \leq 4) \\
&\Leftrightarrow \exists x \in \mathbb{Z}\ \neg (\exists y \in \mathbb{Z} : 3x + y \leq 4) \\
&\Leftrightarrow \exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : \neg (3x + y \leq 4) \\
&\Leftrightarrow \exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : 3x + y > 4
\end{aligned}
$$

If we compare the outcomes of the negation process with the original propositions it turns out that every $\forall$-sign has been replaced by a $\exists$-sign and vice versa and furthermore that the property, in these cases inequalities, has been negated as well: it has become the exact opposite. This holds in general, so negating such a proposition is not a difficult procedure.

Exercise 6 *Negate propositions 2 and 4 of example 3.*

Section 1.5

This section discusses several mathematical proving techniques.

Direct Proof: This is the most straightforward type of proof. It is used to prove that a proposition P is true.

Example 7 *(proposition 3 of Example 3):*

Prove that the proposition $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : 3x + y \leq 4$ is true.

It is extremely important that you read the proposition from the left to the right. So we have to take the x first and then the y. This turns out to be the reason that this proposition is true and proposition 4 in example 3 is false. We will describe a few general concepts with respect to proofs.

1. To prove something for all x is the same as to prove it for an arbitrary x.

   Since we have to prove something for all integers x, we start the proof by taking an arbitrary one. We write this as follows: "Let $x \in \mathbb{Z}$". Then we are allowed to choose a very specific integer y (because it says "there exists a y" and not "for all y" which would have meant that we were not allowed to be specific concerning the y).

2. If we choose a number y after we take x, then we already know the value of x at the moment that we choose the value of y. This means that, if we are allowed to be specific in our choice for the value of y, we may let it depend on the value of x.

We want that $3x + y \leq 4$. This would mean that we need that $y \leq 4 - 3x$. Let's just take $y = 4 - 3x$, which happens to be an integer anyway (of course we can also take $y = 3 - 3x$ or $y = -37 - 3x$, but not $y = -37 - 4x$, since then $3x + y = 3x + (-37 - 4x) = -x - 37$, which is not necessarily ≤ 4). Then what happens to $3x + y$? Well, $3x + y = 3x + (4 - 3x) = 4$ and since $4 \leq 4$ we can now conclude that indeed $3x + y \leq 4$. If we write this down formally, then the proof looks as follows:

Proof. Let $x \in \mathbb{Z}$. Take $y = 4 - 3x \in \mathbb{Z}$. Then

$$3x + y = 3x + (4 - 3x) = 4 \ \ (\leq 4),$$

which completes the proof. ■

Often a proposition is of the form $p \Rightarrow q$. We know that some statement p is true (p could for instance be the proposition that some number x is a natural number) and then we want to prove that some other statement q is true as well (e.g. q could state that $x^2 \geq x$). In this case the proposition as a whole would read: "If x is a natural number, then $x^2 \geq x$", or in mathematical notation:

$$x \in \mathbb{N} \Rightarrow x^2 \geq x$$

Notice that from the first part of the proposition "If x is a natural number" we can conclude that it is about arbitrary natural numbers. Therefore it is about all natural numbers. This means that the proposition can be described in an equivalent manner as follows:

$$\forall x \in \mathbb{N} : x^2 \geq x.$$

If a proposition can be written in more than one way, you can choose which version you want to prove.

To construct a direct proof for a proposition of the form $p \Rightarrow q$, we start by assuming that p is true and we show that as a logical consequence q must be true as well.

Proof. Let $x \in \mathbb{N}$ (this is taking an arbitrary natural number or, in terms of the above mentioned propositions p and q: p is true). Then

$$\begin{aligned} x^2 &= x \cdot x && \text{(definition of } x^2\text{)} \\ &\geq 1 \cdot x && \text{(since } x \text{ is a natural number, which is a positive integer)} \\ &= x, \end{aligned}$$

so apparently $x^2 \geq x$, which is exactly the statement in proposition q. Hence, q is true and this is exactly what we wanted to prove. ■

Examples 5 and 7 show how we can use this fact to prove propositions in different ways.

Example 8 *Prove that for realvalued x we have:*

$$-2 < x < 2 \quad \Rightarrow \quad x^2 + x - 6 < 0$$

(or: $\forall x \in (-2, 2) : x^2 + x - 6 < 0$).

Proof. Let $-2 < x < 2$. Then

$$x^2 < 4 \text{ and } x < 2 \text{ (and } -6 = -6)$$

Therefore $x^2 + x - 6 < 4 + 2 - 6 = 0$, which completes the proof. ■

Example 9 *Prove that the following proposition is true:*

$$\forall x \in \mathbb{R}\, \exists y \in \mathbb{R} : x^2 = y$$

Proof. Let $x \in \mathbb{R}$. Take $y = x^2 \in \mathbb{R}$. Then $x^2 = y$. ■

Exercise 7 *Prove the following propositions:*

1. $\exists x \in \mathbb{N} : x = 3$
2. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : x + y \geq 4$
3. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : y - x = -1$
4. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists z \in \mathbb{R} : x + y = 2z$
5. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R} : (x < y \Rightarrow 2x < 2y)$
6. $\forall x \in \mathbb{R} : (x \geq 10 \Rightarrow x^5 \geq 3x^4 + 5x^2 + 2333)$

Exercise 8 ***(optional)*** *The following exercises are fairly tricky. They are optional for those who want to give them a go. They are ordered by increasing degree of difficulty: The first exercise is fairly straightforward, the second one starts off the same, but then requires an extra argument, the third one uses a slightly trickier argument and the fourth one is a more difficult version of the third one.*

1. $\forall n \in \mathbb{N} : n$ *odd* $\quad\Rightarrow\quad n^2 - 1$ *is divisible by 8*
2. $\forall n \in \mathbb{N} : (n$ *odd and* n *is not divisible by 3*$) \quad\Rightarrow\quad n^2 - 1$ *is divisible by 6*
3. $\forall n \in \mathbb{N} : n^5 - n$ *is divisible by 30*
4. $\forall n \in \mathbb{N} : n^7 - n$ *is divisible by 42*

Contrapositive Proof: We have seen that the proposition $\neg q \Rightarrow \neg p$ is equivalent to the proposition $p \Rightarrow q$. A contrapositive proof uses that knowledge: Instead of proving that the proposition $p \Rightarrow q$ is true we prove that $\neg q \Rightarrow \neg p$ is true.

Example 10 *We prove example 5 in a contrapositive way.*

Proof. The first step is to write the contrapositive of the proposition $-2 < x < 2 \quad \Rightarrow x^2 + x - 6 < 0$. This is the following proposition: For realvalued x we have:

$$x^2 + x - 6 \geq 0 \quad \Rightarrow \quad x \leq -2 \text{ or } x \geq 2$$

Let $x \in \mathbb{R}$ such that $x^2 + x - 6 \geq 0$. Since

$$x^2 + x - 6 = (x - 2)(x + 3)$$

this implies that either

i) $x - 2$ and $x + 3$ are both ≥ 0, so $x \geq 2$ or

ii) $x - 2$ and $x + 3$ are both ≤ 0, which means that $x \leq -3$

So $x \geq 2$ or $x \leq -3$, which means that also $x \geq 2$ or $x \leq -2$, which proves the proposition. ■

We have seen that the propositions $x \in \mathbb{N} \Rightarrow x^2 \geq x$ and $\forall x \in \mathbb{N} : x^2 \geq x$ are equivalent. More generally we have: For any set A and any property $\mathbb{P}$ that elements out of A may or may not possess the following two propositions are equivalent:

$$x \in A \Rightarrow \mathbb{P}(x) \text{ and}$$
$$\forall x \in A : \mathbb{P}(x)$$

The left propositions reads: If x is in the set A, then x must have property $\mathbb{P}$; the right one: For all x in A we have property $\mathbb{P}$. This means that we can use a contrapositive proof to prove the proposition $\forall x \in A : \mathbb{P}(x)$: Instead we prove $\neg\mathbb{P}(x) \Rightarrow x \notin A$. Sometimes this may be easier than finding a direct proof.

Example 11 *Prove that* $\forall x \in \mathbb{Z} : x^2 \neq 3$.

Proof. We use the contrapositive and prove that $x^2 = 3 \Rightarrow x \notin \mathbb{Z}$. Let $x^2 = 3$. Then $x = \sqrt{3}$ or $x = -\sqrt{3}$. In either case $x \notin \mathbb{Z}$. ■

Exercise 9 *Prove the following propositions using a contrapositive proof:*

1 $\forall x \in \mathbb{R} :\ x^2 - x - 6 \neq 0 \ \Rightarrow\ x \notin \{-2, 3\}$

2 For all nonnegative real numbers x and y we have: $3x + 4y < 12 \ \Rightarrow\ (x < 4) \wedge (y < 3)$

3 $\forall x > 0\ \forall y > 0 : x^2 + y^2 > 1 \Rightarrow x + y > 1$

Exercise 10 *Prove the following proposition: Let a and b be real numbers. Then* $(\forall \varepsilon > 0 : b < a + \varepsilon) \Rightarrow b \leq a$.

This proposition says that if b is smaller than all numbers that are bigger than a, then b can not be bigger than a (or: b is smaller than or equal to a). The greek letter ε is in mathematics very often used to denote a number that is a very tiny little bit bigger than 0.

Biconditional Proof: To prove an equivalency: $p \Leftrightarrow q$. In such a case both $p \Rightarrow q$ and $p \Longleftarrow q$ have to be proved. This means that we have to prove two propositions at once!

Example 12 *Show that for all natural numbers x and y we have:* $x \geq y + 1 \Leftrightarrow x^2 \geq y^2 + 3$.

Proof. First we prove the proposition where the implication arrow points from left to right. We write "$\Rightarrow$":

Let $x \geq y + 1$. Then $x^2 \geq (y + 1)^2$ (notice that we deal with natural numbers here and they are always positive!). So

$$\begin{aligned} x^2 &\geq (y+1)^2 \\ &= y^2 + 2y + 1 \\ &\geq y^2 + 2 \cdot 1 + 1 \text{ (since } y \geq 1) \\ &= y^2 + 3 \end{aligned}$$

which is exactly what we needed.

"$\Leftarrow$":

Let $x^2 \geq y^2 + 3$. Then x must be bigger than y and, since they are both natural numbers, this implies that $x \geq y + 1$. So now we have proved the biconditional proposition. Such a proposition is often also called an **equivalency**, because it claims that two smaller propositions are equivalent. ■

Exercise 11 *Prove that for all real numbers x we have: $\left(x^2 + 5x - 6 = 0\right) \Leftrightarrow (x = -6 \vee x = 1)$.*

Exercise 12 *Prove the following proposition:*

$$\forall m \in \mathbb{N}\, \forall n \in \mathbb{N}: \quad m \cdot n \textit{ is odd} \Leftrightarrow m \textit{ and } n \textit{ are odd}$$

Proof by Contradiction: If we apply a proof by contradiction, then instead of proving that a proposition P is true, we prove that $\neg P$ is false. We do so by assuming that $\neg P$ is true and applying logical consequences to arrive at a contradiction. In the book a proof by contradiction is used to prove that $\sqrt{2}$ is not a rational number.

Example 13 *Prove that in a right-angled triangle the length of the hypothenuse (c) is smaller than the sum of the lengths of the other two sides (a and b).*

Proof. The proof uses contradiction and relies on the Pythogarean theorem that states that for such a triangle we have $a^2 + b^2 = c^2$. So, we assume that the statement in the example is false, so $c > a + b$. Then by squaring both sides (and noticing that all lengths are positive) we find

$$c^2 > (a + b)^2. \tag{1}$$

But

$$(a + b)^2 = a^2 + 2ab + b^2 > a^2 + b^2. \tag{2}$$

Combining (1) and (2) gives $c^2 > a^2 + b^2$, which contradicts the Pythagorean theorem. We conclude that $c \leq a + b$. ■

Exercise 13 *Let S be the set consisting of all real numbers that are smaller than 1, (i.e. $S = \{x \in \mathbb{R} : x < 1\}$). Prove that the set S has no maximum (i.e. there is no largest number in S). In layman's terms this statement says that there is no largest number that is smaller than 1.*

Proof by Counterexample: A proposition that starts with a $\forall$-sign claims that something is true for all instances. Such a proposition would therefore be false if you can find only one instance for which this something is NOT true. Providing such an instance is called providing a counterexample and it would prove that the proposition is false (or: it **disproves** the proposition).

Example 14 *Prove that the following proposition is false:*

$$\forall x \in \mathbb{N}\, \exists y \in \mathbb{N} : y - x = -1$$

Proof. Take $x = 1$. Then for all $y \in \mathbb{N}$ we have that $y \geq x$ and hence $y - x \neq -1$, which disproves the proposition. ■

Notice that we already proved that the same proposition is true if x and y are real numbers (cf. exercise 7). This shows that you have to be careful in checking the sets that the numbers belong to.

Example 15 *(proposition 4 in example 3): Disprove the following proposition:*

$$\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : 3x + y \leq 4$$

Proof. Let $x \in \mathbb{Z}$ and take $y = -3x + 5 \in \mathbb{Z}$. Then $3x + y = 3x + (-3x + 5) = 5 \nleq 4$, which disproves the proposition. ■

Exercise 14 *(a more formal description of exercises 1.5.2 (a) and (b)): Disprove the following propositions by means of a counterexample:*

a For all prime numbers p and q we have: If $n = p^2 + q^2$, then n is prime.

b $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R} : \quad a > b \Leftrightarrow a^2 > b^2$.

Exercise 15 *(several exercises from section 1.5 and more): Prove or disprove:*

1. $\sqrt{5}$ is a rational number
2. $\forall x \in \mathbb{R} : x^4 = 1 \Rightarrow x = 1$
3. $\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : x + y$ is even $\Leftrightarrow$ x and y are both even or x and y are both odd
4. $\forall x \in \mathbb{R} : x^2 - 4 < 0 \Rightarrow -2 < x < 2$
5. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \exists z \in \mathbb{N} : x + y = 2z$
6. $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R} : x^2 = y$
7. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N} : (x < y \ \Rightarrow\ x^2 < y^2)$
8. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R} : (x < y \ \Rightarrow\ x^2 < y^2)$
9. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \forall z \in \mathbb{N}$: $((x < y) \wedge (y < z) \ \Rightarrow\ xy < yz)$
10. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \forall z \in \mathbb{R}$: $((x < y) \wedge (y < z) \ \Rightarrow\ xy < yz)$
11. $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N} : |x| \leq \frac{1}{2}|y|$
12. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R} : |x| \leq \frac{1}{2}|y|$
13. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N} : (x^2 + y^2 = 1 \ \Rightarrow\ x + y = 1)$
14. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R} : (x^2 + y^2 = 1 \ \Rightarrow\ x + y = 1)$
15. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N} : (x^2 - y^2 = 1 \ \Rightarrow\ x = 2)$

16. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R} : (x^2 - y^2 = 1 \quad \Rightarrow \quad x = 2)$

17. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R} : (x^2 - y^2 = 1 \quad \Rightarrow \quad x = 2)$

18. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \exists z \in \mathbb{N} :\ x^2 + y^2 = z^2$

19. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \exists z \in \mathbb{R} :\ x^2 + y^2 = z^2$

Section 1.6

Mathematical Induction:

Pierre de Fermat was a French mathematician from the 17^{th} century, who is famous for his 'Last Theorem'. It states that the equation $x^n + y^n = z^n$ does not have any integer-valued solutions if n is a natural number ≥ 3. This theorem used to be called Fermat's conjecture until it was finally solved in 1994 by Andrew Wiles. Fermat was also interested in prime numbers and at some stage he figured out that the numbers

$$\begin{aligned}
2^{2^0} + 1 &= 3 \\
2^{2^1} + 1 &= 5 \\
2^{2^2} + 1 &= 17 \\
2^{2^3} + 1 &= 257 \\
2^{2^4} + 1 &= 65537
\end{aligned}$$

were all primes. He then conjectured that $2^{2^5} + 1, 2^{2^6} + 1, 2^{2^7} + 1, \ldots$ are primes. However, he couldn't manage the calculations. (This is kind of understandable, given that $2^{2^5} + 1 = 4294967397$.) One hundred years later Leonard Euler proved that $2^{2^5} + 1$ is divisible by 641, disproving the conjecture. So even if it looks like there is a pattern, this may completely disappear when we add more data. This mistake of finding nonexistent patterns, is, thanks to a humorous paper by Richard Guy, often referred to as the 'Strong Law of Small Numbers'. One more funny fact about Fermat's mistake: It has been proved that actually $2^{2^n} + 1$ is never a prime number for $n \geq 5$, so he couldn't really have been more wrong.

Example 16 *Consider the following sequence of numbers:* $5^1 + 3, 5^2 + 3, 5^3 + 3, 5^4 + 3, \ldots$. *We want to know if every number in this sequence is divisible by 4. Doing a few calculations we find that*

$$\begin{aligned}
5^1 + 3 &= 8 \\
5^2 + 3 &= 28 \\
5^3 + 3 &= 128 \\
5^4 + 3 &= 628
\end{aligned}$$

are all divisible by 4. Fermat's mistake tells us that we can not really draw any conclusions from this. Furthermore, do you really want to calculate $5^5 + 3$, $5^6 + 3$, $5^7 + 3$, *etc. and notice* *that they are all divisible by 4? Let's apply a more systematic procedure:*

$$\begin{aligned}
5^5+3 = 5\cdot 5^4+3 &= 4\cdot 5^4 + (5^4+3) = \textit{multiple of 4 + multiple of 4} &&= \textit{multiple of 4}\\
5^6+3 = 5\cdot 5^5+3 &= 4\cdot 5^5 + (5^5+3) = \textit{multiple of 4 + multiple of 4} &&= \textit{multiple of 4}\\
5^7+3 = 5\cdot 5^6+3 &= 4\cdot 5^6 + (5^6+3) = \textit{multiple of 4 + multiple of 4} &&= \textit{multiple of 4}\\
5^8+3 = 5\cdot 5^7+3 &= 4\cdot 5^7 + (5^7+3) = \ldots \textit{(etc.)}
\end{aligned}$$

Each time we used the fact that we just figured out that the previous number was a multiple of 4 to conclude that the next number is also divisible by 4. This is the general idea of a proof by mathematical induction.

In example 16 we have a collection of propositions $P(n)$; one proposition for each natural number n. $P(1)$ is the proposition that states that 5^1+3 is divisible by 4, $P(2)$ states that 5^2+3 is divisible by 4, $P(3)$ states that 5^3+3 is divisible by 4, etc. and then we have an overall proposition P that states that $P(n)$ is true for all n. Mathematical induction is a two-step procedure to prove that P is true.

1. Prove that $P(1)$ is true. Is step is called the **Basic Step**, or **Step 0** and it usually boils down to plugging in $n = 1$ in the proposition and noticing that the resulting proposition is true.
   In example 16 the proposition $P(1)$ states that $5^1 + 3$ is divisible by 4, which is obviously true. Be careful: sometimes the smallest number may not be $n = 1$; it could also be $n = 0$ or $n = 2$ or another number. But the idea of the basic step remains the same: We prove that the proposition $P(n)$ is true for the smallest value of n.

2. Prove that $P(n) \Rightarrow P(n+1)$ for all $n \geq 1$ (or $n \geq 0$ or $n \geq 2$, depending on what the smallest value of n was). This step is called the **Inductive Step**, or **Step 1**.

It is important to realize that the two steps together form the entire proof: there is nothing more that has to be done. In example 16 you can see why that is the case. If $P(1)$ is true ($5^1 + 3$ divisible by 4), then we can figure out from there that $P(2)$ is true, which in turn can be used to show that $P(3)$ is true etc.

The following example is arguably the most classic example of a proof by mathematical induction.

Example 17 *Show that for each natural number n we have*

$$1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1)$$

Proof. We use mathematical induction, where, for all $n \in \mathbb{N}$, $P(n)$ is the proposition: $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$.

Step 0: (Prove $P(1)$). The proposition $P(1)$ states that $1 = \frac{1}{2}\cdot 1 \cdot 2$, which is (obviously) true.

Step 1: (Prove that for all $n \geq 1$ we have: $P(n) \Rightarrow P(n+1)$). This is an implication; we have to show that, given that $P(n)$ is true, the proposition $P(n+1)$ is also true. Let's first write down the proposition $P(n+1)$. It is an equation whose left hand side reads as follows:

$1 + 2 + 3 + \ldots + n + (n + 1)$. This has to do with the fact that the left hand side of $P(n)$ is a sum of n terms; to construct the left hand side of $P(n + 1)$ we need to add the next term to this sum, in this case $(n + 1)$. The right hand side of $P(n)$ is a product of two factors that depend on n. This means that the right hand side of $P(n + 1)$ should be the same product, where n is replaced by $n + 1$. Hence it becomes $\frac{1}{2}(n + 1)((n + 1) + 1) = \frac{1}{2}(n + 1)(n + 2)$. So apparently the proposition $P(n+1)$ states that $1+2+3+\ldots+n+(n+1) = \frac{1}{2}(n+1)(n+2)$. We can now start step 1 of the induction proof:

Assume that $P(n)$ is true. Then $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$ (this is called the **induction hypothesis**). But then

$$\begin{aligned} 1 + 2 + 3 + \ldots + n + (n + 1) &= (1 + 2 + 3 + \ldots + n) + (n + 1) \\ &= \tfrac{1}{2}n(n + 1) + (n + 1) \qquad \text{(by the induction hypothesis)} \\ &= (n + 1) \cdot (\tfrac{1}{2}n + 1) \\ &= \tfrac{1}{2}(n + 1)(n + 2) \end{aligned}$$

hence $P(n + 1)$ is true, which completes the proof. ■

Exercise 16 *Show that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.*

The proposition $P(n)$ does not necessarily consist of an equation and the smallest possible value of n is not necessarily $n = 1$. In example 18 we prove a collection of inequalities, where the smallest number for which the inequality holds, is $n = 10$.

Example 18 *Show that for all natural numbers $n \geq 10$ we have: $2^n \geq n^3$, or:*

$$\forall n \in \mathbb{N} : \qquad n \geq 10 \;\Rightarrow\; 2^n \geq n^3.$$

Proof. By induction, where $P(n)$ is the proposition: $2^n \geq n^3$.

Notice that for $n = 9$ we have $2^9 = 512 < 729 = 9^3$, so for values of n smaller than 10 indeed the proposition $P(n)$ is not always true.

Step 0: The basic step this time is for $n = 10$, since that is the smallest value of n for which the proposition needs to be proved. We have $2^{10} = 1024 \geq 1000 = 10^3$.

Step 1: Let $n \geq 10$ and suppose that $P(n)$ is true. Then $2^n \geq n^3$. But then

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\geq 2 \cdot n^3 \text{ (by the induction hypothesis)} \\ &= n^3 + n^3 \\ &= n^3 + n \cdot n^2 \\ &\geq n^3 + 10n^2 \text{ (since } n \geq 10) \\ &= n^3 + 3n^2 + 7n^2 \\ &= n^3 + 3n^2 + 7 \cdot n \cdot n \\ &\geq n^3 + 3n^2 + 70n \text{ (since } n \geq 10) \\ &= n^3 + 3n^2 + 3n + 67n \\ &> n^3 + 3n^2 + 3n + 1 \\ &= (n+1)^3 \, , \end{aligned}$$

which completes the proof. ■

Exercise 17 *Exercises 2 & 3 from section 1.6.*

Exercise 18 *Prove the following propositions by mathematical induction:*

1. $\forall n \in \mathbb{N} : 7^n - 1$ *is divisible by 6*
2. $\forall n \in \mathbb{N} : 7^n + 2n - 1$ *is divisible by 4*
3. $\forall n \in \mathbb{N} : 6^{2n-1} + 5^{2n-1}$ *is divisible by 11*
4. $\forall n \in \mathbb{N} : 4^{2n-1} + 3^{n+1}$ *is divisible by 13*
5. $\forall n \in \mathbb{N} : 7^n + 3^n - 2$ *is divisible by 8 (a bit trickier!)*
6. $\forall n \in \mathbb{N} : 1 + 3 + 5 + \ldots + (2n - 1) = n^2$
7. $\forall n \in \mathbb{N} : 5 + 8 + 11 + \ldots + (3n + 2) = \frac{1}{2}n(3n + 7)$
8. $\forall n \in \mathbb{N} : 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots + n \cdot (n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$
9. $\forall n \in \mathbb{N} : 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \ldots + n \cdot (n + 3) = \frac{1}{3}n(n + 1)(n + 5)$
10. $\forall n \in \mathbb{N} : 2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \frac{2}{3}n(n + 1)(2n + 1)$
11. $\forall n \in \mathbb{N} : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n \cdot (n+1)} = 1 - \frac{1}{n+1}$
12. $\forall n \in \mathbb{N} : \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

We conclude chapter 1 by proving some of the biggest theorems in mathematics. The first one is the so-called *Fundamental Theorem of Arithmetic*. It says:

Theorem 1 *Every natural number can be expressed as a product of prime factors in a unique way.*

Proof. We will prove only the existence part here; the uniqueness part will be omitted. Notice first that 1 is the product of 0 prime factors (such a product is called an empty product), so it does indeed satisfy the statement in the theorem. For $n \geq 2$ we will prove the existence-part by induction. So, let $P(n)$ be the proposition that states that the number n can be expressed as a product of prime factors.

Basic step: $2 = 2$, so indeed 2 is a (one-factor) product of prime numbers and $P(2)$ is true.

Inductive step: Instead of proving $P(n) \Rightarrow P(n+1)$ we prove $P(2) \wedge P(3) \wedge P(4) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$. Notice that this is essentially the same thing. So, assume that all numbers 2, 3, 4,...,n can be expressed as a product of prime factors. What can we say about $n+1$? There are two cases:
Case 1: $n+1$ is prime. In this case $n+1 = n+1$, so $n+1$ is a one-factor product of prime numbers and $P(n+1)$ is true.
Case 2: $n+1$ is not prime. In this case $n+1 = p \cdot q$ for some natural numbers $p \geq 2$ and $q \geq 2$. We know that p and q are both smaller than $n+1$, so according to the induction hypothesis, both p and q can be expressed as a product of prime factors, say $p = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_k^{l_k}$ and $q = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_k^{m_k}$, where p_k, the k^{th} prime number, is the highest one appearing in p and/or q. So, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ etc. Notice that many of the exponents may be equal to 0. Now we have: $n+1 = p_1^{l_1+m_1} \cdot p_2^{l_2+m_2} \cdot \ldots \cdot p_k^{l_k+m_k}$, so apparently $n+1$ can also be expressed as a product of prime factors and $P(n+1)$ is true, which completes the proof. ■

It is essentially 'thanks to' the fundamental theorem of arithmetic that the number 1 is not a prime. It used to be considered a prime number. However, 1 being prime destroyed the uniqueness part of the fundamental theorem of arithmetic (you can multiply by 1 any number of times). Mathematicians really like uniqueness in a theorem; it makes it much stronger. Hence at some stage the decision was made that 1 would no longer be a prime number (it used to be considered prime) in favour of having a stronger theorem. We will now use it to prove another famous mathematical theorem.

Theorem 2 *There is an infinite number of primes.*

Proof. We prove this theorem by contradiction. So, we start by assuming that the statement in the theorem is false: We aasume that there is a finite number of primes, say n. Then, like in the proof of the fundamental theorem of arithmetic, there must be a biggest one, say p_n and the complete set of primes looks as follows:

$$\{p_1, p_2, p_3, \ldots, p_n\} = \{2, 3, 5, \ldots, p_n\}.$$

Now we consider the following number:

$$N = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n + 1 = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_n + 1$$

so N is the product of all prime numbers plus 1. Then N is not divisible by 2, not divisible by 3, not divisible by 5, not divisible by 7,... and, eventually, not divisible by p_n. But this means that N is not divisible by any of the prime numbers. However, according to the fundamental theorem of arithmetic, every natural number can be written as a product of prime factors. This means that N, which can not be written as a product of the numbers $\{p_1, p_2, p_3, \ldots, p_n\}$ must be a prime number itself! This, however, contradicts our assumption than p_n is the largest prime (notice that N is a lot bigger than p_n!). This completes the proof by contradiction and we conclude that the number of primes must be infinite! ■

Exercise 19 *Use the fundamental theorem of arithmetic to prove that in a (perfect) square all prime factors appear an even number of times.*

Extra section: Common mistakes in proofs!

If the proposition $p \Rightarrow q$ is true, this does not automatically mean that the proposition p is true (or formally: $(p \Rightarrow q) \not\Rightarrow p$). In fact, as long as q is true then $p \Rightarrow q$ will always be true, even if p is false. The fact that q is true does not mean anything!! Below you can find an example, where after every line I have put my thoughts about that line in between braces. At the end of the example you can find a more detailed explanation of what went wrong.

Example 19 *We 'prove' that* $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : x + y \geq 4$.

Proof. Let $x \in \mathbb{R}$. Take $y = 4 - x \in \mathbb{R}$ (so far everything is fine). Then

$$\begin{aligned} x + y &\geq 4 \text{ (This is too quick,)} \\ &\quad \text{(it should have been the final conclusion.)} \\ \text{so } x + 4 - x &\geq 4 \text{ (Oh no...)} \\ \text{so } 4 &\geq 4 \text{ (Is \textit{that} the conclusion?)} \end{aligned}$$

which completes the 'proof'. ■

So the final conclusion is that $4 \geq 4$; we have just 'proved' that $4 \geq 4$. That's not really a new result, is it? This obviously was not what was supposed to be proved. At the same time there was an assumption made in the second line of the proof, namely that $x + y \geq 4$, which is exactly the statement that had to be proved. So where did we go wrong? That is actually quite simple: we did not really prove that $x + y \geq 4$, we started our line of reasoning there instead of finishing there.

Let's put the above "proof" in the context of section 1.3: Let p be the proposition that states that $x+y \geq 4$ and let q be the proposition stating that $4 \geq 4$. Then we have just proved $p \Rightarrow q$. This is not very impressive though, since obviously q is true and therefore $p \Rightarrow q$ is always true (check this in section 1.3!). What we did *not* do, is proving that p is true, and that was the question.

Now an example of how not to use mathematical induction. Here there are two different types of mistakes (excluding calculation errors) that are made with some regularity. Both of them will be addressed.

Example 20 *(This is actually Example 13, but then with a faulty proof) Show that for each natural number n we have*

$$1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1)$$

Proof.

Step 0: $P(1) = 1 = \frac{1}{2} \cdot 1 \cdot 2$.

Here we have the first common mistake: $P(1)$ is not a number; it is a *proposition* stating that $1 = \frac{1}{2} \cdot 1 \cdot 2$. Since indeed $1 = \frac{1}{2} \cdot 1 \cdot 2$ we conclude that $P(1)$ is true.

Step 1: Let $P(n)$ be true.

$$\Rightarrow \quad 1+2+3+\ldots+n+(n+1)=\tfrac{1}{2}(n+1)(n+2)$$

(This is the statement that had to be proved, don't start there!!)

$$\Rightarrow \quad \tfrac{1}{2}n(n+1)+(n+1)=\tfrac{1}{2}(n+1)(n+2)$$

(The final conclusion will not make sense.)

$$\Rightarrow \quad \tfrac{1}{2}n^2+\tfrac{1}{2}n+n+1=\tfrac{1}{2}(n^2+3n+2)$$

$$\Rightarrow \quad \tfrac{1}{2}n^2+\tfrac{3}{2}n+1=\tfrac{1}{2}n^2+\tfrac{3}{2}n+1$$

(Indeed a nonsensical conclusion!)

The second common mistake: It is essentially the same mistake as in the previous example: We started by assuming the statement that you had to prove, and then we arrived at a conclusion that does not make sense. Obviously that was bound to happen, because after we assumed in the first line that $1+2+3+\ldots+n+(n+1)=\frac{1}{2}(n+1)(n+2)$, there was nothing left to prove. This mistake is by far the most common structural mistake that is ever made in these types of exercises! ■

Exercise 20 *Give propositions p and q such that step 1 of the faulty proof above corresponds to the proof of $p \Rightarrow q$.*

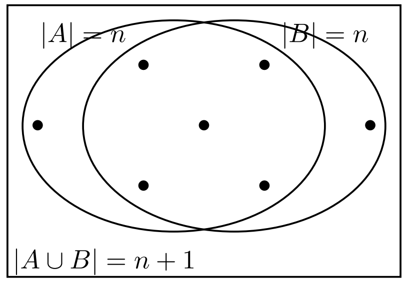
When you try to prove something by mathematical induction, you have to be very careful. It is possible to "prove" obviously false propositions by making a tiny mistake in an assumption or a calculation. Making faulty calculations often means accidentally dividing by 0; when we cancel out a common factor on the left hand side and on the right hand side of an equation, we have to make sure that this factor was not equal to 0. Making faulty assumptions may be a bit more complicated. In the following two exercises we 'prove' something that is obviously false. Your task is to find the (reasonably well-hidden) flaw in the proof.

Exercise 21 *In a group of people of size $n \in \mathbb{N}$ either all people are male or all people are female.*

Proof. (by mathematical induction to the size of the group)

Basic step: For a group size of 1 it is obviously true that all people in the group are either male or female.

Inductive step: Let the claim be true for n. Then in any group of size n either all people are male or all people are female. Now consider a group of size $n+1$. If we leave out one person, then we are left with a group of size n which, by the induction hypothesis, consists of only males or only females. Of course we could have taken any subset of size n out of those $n+1$ people, each time being left with a group of size n that, by the induction hypothesis, consists of only males or only females. The following diagram shows the situation schematically:



Here A is a group of size n and B is a group of size n. The notation that we will use for that in chapter 2 is $|A| = n$ and $|B| = n$. The set $A \cup B$ (the union of A and B, which is the set consisting of all elements that are in A or in B; this will also be introduced in chapter 2) has size $n + 1$. We can conclude that the $n + 1$ people in the set $A \cup B$ must either be all male or all female. ■

Exercise 22 *For every pair of natural numbers x and y we have: $x = y$ (or, equivalently: All natural numbers are equal).*

Proof. (by mathematical induction on the maximum of x and y)

Let $\max(x, y) = n$ (where max denotes the value of the biggest number of the 2). We shall proceed by induction on n.
Basic step: If $\max(x, y) = 1$ for natural numbers x and y, then obviously $x = y$, since they are both equal to 1.

Inductive step: Suppose that $\max(x, y) = n + 1$ and consider the numbers $x' = x - 1$ and $y' = y - 1$. Then obviously we have $\max(x', y') = n$. Now, by the induction hypothesis, we must have $x' = y'$, from which it follows that $x' + 1 = y' + 1$ and hence $x = y$ as claimed.

Therefore, all natural numbers are equal. ■

Obviously, at an exam you will never be asked to use induction to prove a proposition that is actually false.