Relations: overview

Definition: a relation R on a set A is a subset of $A \times A$

Three fundamental properties: a relation R on a set A is

- reflexive if $\forall x \in A : xRx$
  - Not reflexive if $\exists x \in A : x \not R x$
- Symmetric if $\forall x, y \in A : xRy \rightarrow yRx$
  - Not symmetric if $\exists x, y \in A : xRy \wedge y \not R x$
- transitive if $\forall x, y, z \in A : (xRy \wedge yRz) \rightarrow xRz$
  - Not transitive if $\exists x, y, z \in A : xRy \wedge yRz \wedge x \not R z$

Equivalence relation R on A:

- Symmetric, transitive, reflexive
- Induces a partition of A, subsets = equivalence classes
- In an equivalence class, all elements are related to each other

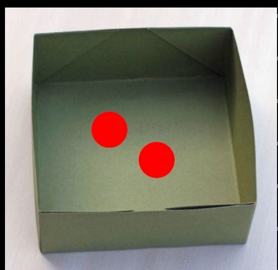
Today

- Anti-symmetric relations
- Partial order

- Intro to functions
- Composition of functions

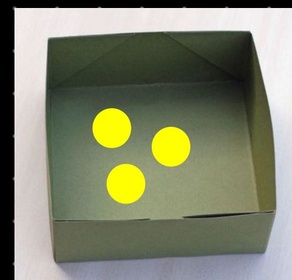
Book: chapter 3, sections 3.3, 3.4, 3.5, 3.7

Recap



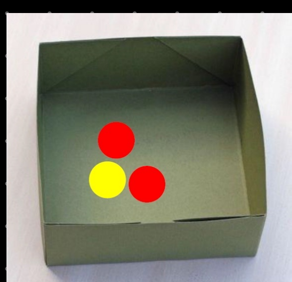
All balls in the box are red

Negation: there is a ball in the box that is not red.



All balls in the box are yellow

Negation: there is a ball in the box that is not yellow.



Not all balls in the box are red and not all balls in the box are yellow.

There is at least one ball that is not red.
There is at least one ball that is not yellow.

All balls in the box are red and all balls in the box are yellow (since there are no balls in the box...)

Anti-symmetry:

NOT the negation of symmetry

- An arrow between different elements can only go in 1 direction.

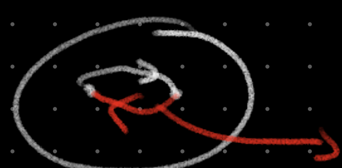
example : xRy if $x \leq y$, on $\mathbb{R}$

xRy if x is a multiple of y, on $\mathbb{N}$

xRy if $x = y + 10$

Definition : $\forall x, y \in A : (xRy \wedge yRx) \rightarrow x = y$

$\forall x, y \in A : (xRy \wedge x \neq y) \rightarrow y \not R x$

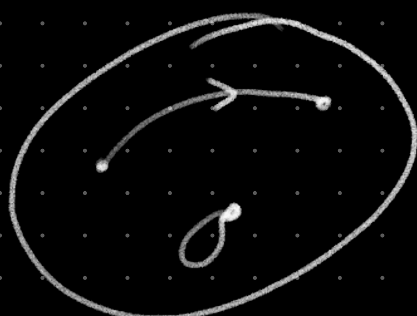


NOT A.S.

Symmetry vs. Anti-symmetry

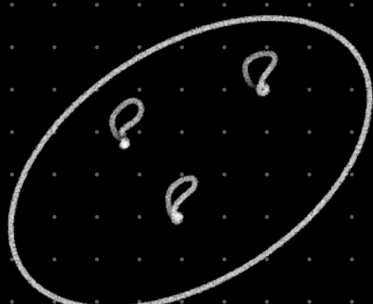
Symmetry : all arrows go in 2 directions

Anti-symmetry : arrows between different elements cannot go in both directions

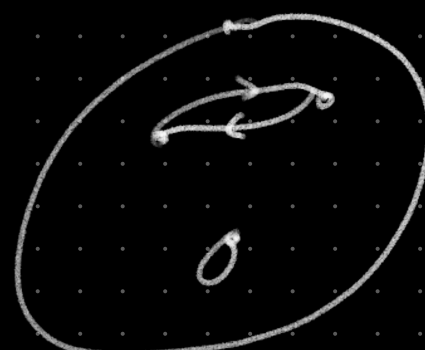


AS ✓
S ✗
T ✓
R ✗

S ✗
AS ✗
T ✗
R ✗



AS ✓
S ✓
T ✓
R ✓



S ✓
AS ✗
T ✗
R ✗

Examples

- xRy if $x \leq y+10$, on $\mathbb{N}$

  $x=1$, $y=20$. Then xRy, since $1 \leq 20+10$
  $y \not R x$, since $20 \not\leq 1+10$

  NOT SYMMETRIC

  $x=1$, $y=2$. Then xRy, since $1 \leq 2+10$
  and $1 \neq 2$ $\quad$ yRx, since $2 \leq 1+10$

  $\longrightarrow$ NOT ANTI-SYMMETRIC

- xRy if $x = y+10$, on $\mathbb{N}$

  assume xRy and $x \neq y$
  then $x = y+10 \Rightarrow y = x-10 \neq x+10$
  so $y \not R x$

  ANTI-SYMMETRIC

Partial order

A relation that is reflexive, transitive and anti-symmetric is called a partial order

example : $\subseteq$, on $\mathbb{P}(\mathbb{N})$

* reflexive : for all set $A \in \mathbb{P}(\mathbb{N})$: $A \subseteq A$

  every set is a subset of itself.

* anti-symmetry : if $A \subseteq B$ and $B \subseteq A$, then $A = B$

  (definition of set equality!)

* transitive : if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

  Let $x \in A$. Then $x \in B$, since $A \subseteq B$.
  Then $x \in C$, since $B \subseteq C$.

$\emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \ldots$

$\{1,3\}$?

xRy if $x \leq y$, on $\mathbb{R}$

- $x \leq x$ true → reflexive
- if $x \leq y$ and $y \leq x$, then $x = y$ → anti-sym
- $x \leq y$ and $y \leq z$ → $x \leq z$ → transitive.

- xRy if x is a multiple of y, on $\mathbb{N}$
  - reflexive? xRx if x is a multiple of x

    → true for all $x \in \mathbb{N}$.
  - transitive? $xRy \wedge yRz \rightarrow xRz$

    xRy, so $x = ky$ $(k \in \mathbb{N})$

    yRz, so $y = mz$ $(m \in \mathbb{N})$

    then $x = k(mz) = (km)z$, so xRz
  - anti-sym? $xRy \wedge yRx$

    $\Rightarrow x = ky \wedge y = mx \Rightarrow x = km \cdot x$

    $\Rightarrow k = m = 1 \Rightarrow y = x$

Partial and total order

transitive
anti-symmetric
reflexive

partial order

$+ \ \forall x,y \in A: xRy \vee yRx$

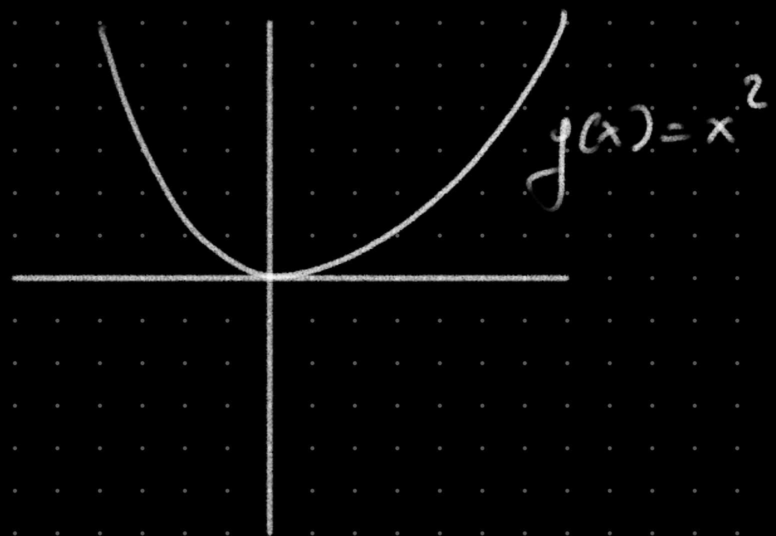
NOT EXAM MATERIAL :)

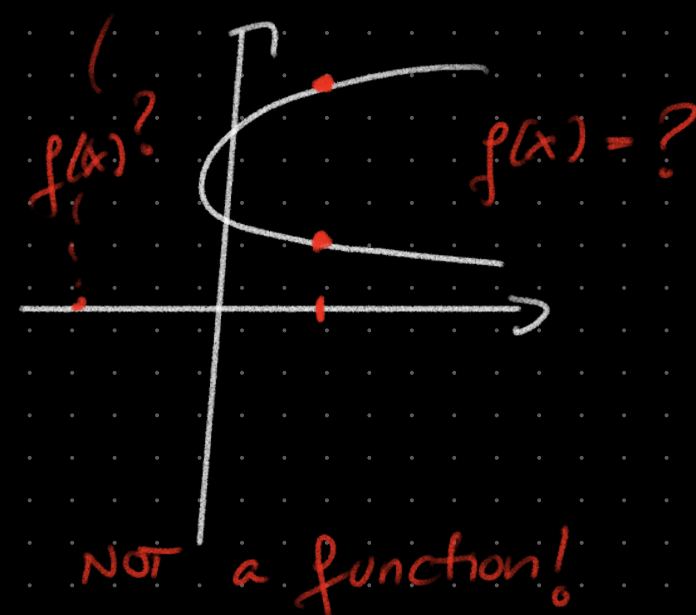
(↳ this slide only)

Functions

A function f is a mapping from a set A to a set B. To be a function, f(x) has a UNIQUE value for EVERY element x of A.

- Notation: f: A->B where f(x)=...

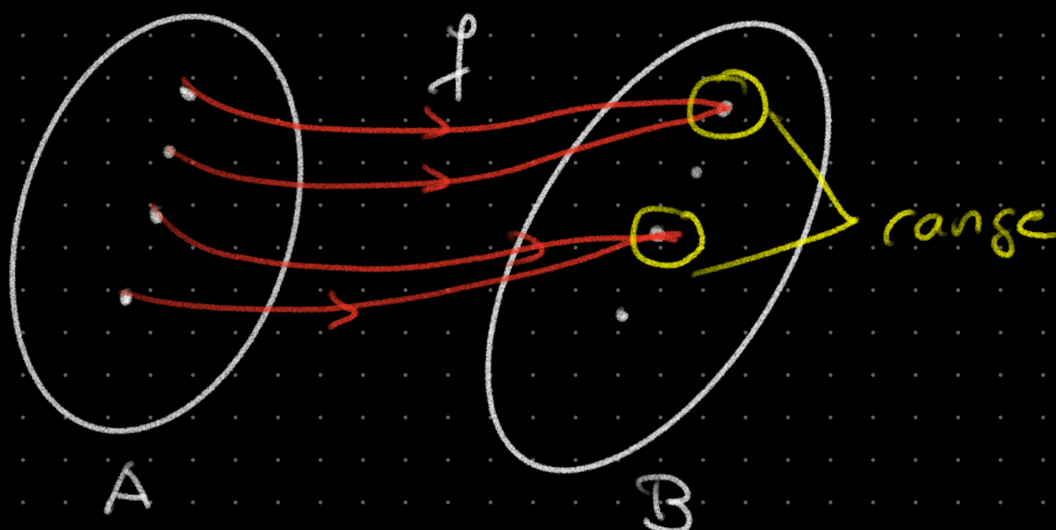




$f : A \to B$, where $f(x) = \cdots$

domain   co-domain

$f(x)$ is UNIQUELY defined for EVERY $x \in A$



- the RANGE of a function : the set of elements in B that are reached by f.

$$\text{range}(f) = \{ y \in B \ \exists x \in A : y = f(x) \}$$

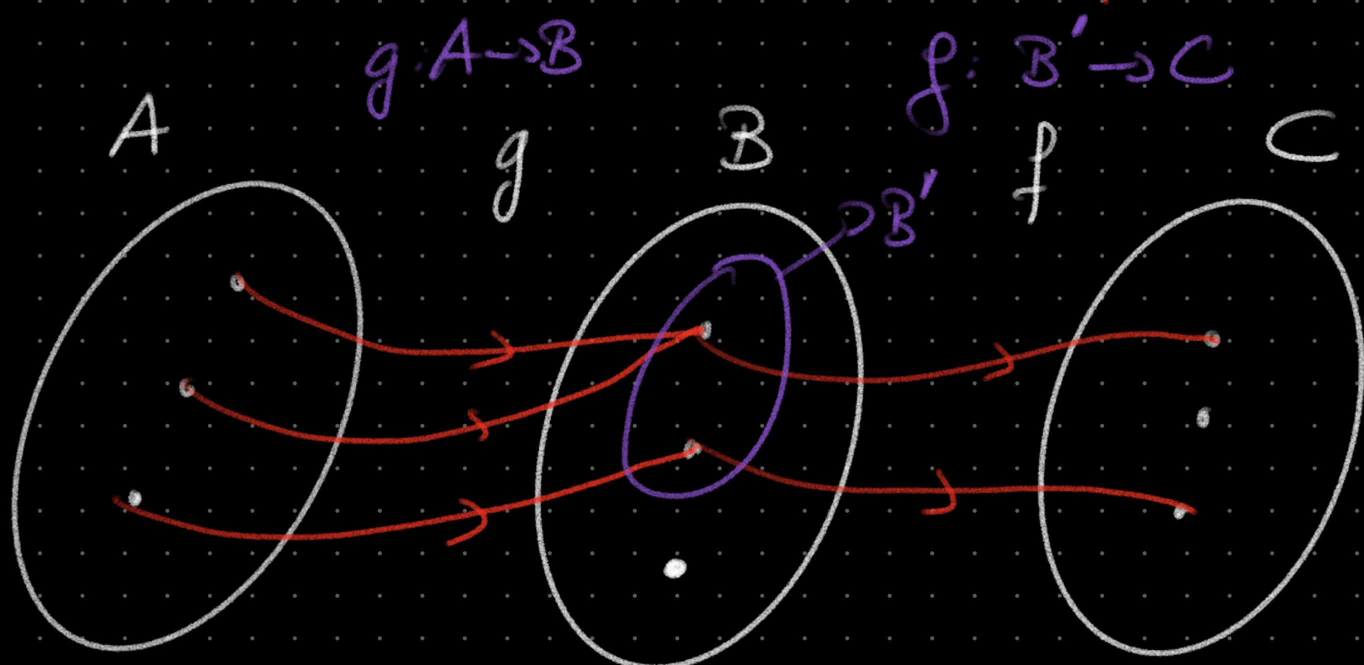
$$\text{range}(f) \subseteq \text{co-domain}(f)$$

Composition of functions

Consider functions g: A->B and f: B->C.

Then (f o g)(x) = f(g(x)) is the function obtained first using g and then using f.

- Domain of (f o g) = domain of g
- Co-domain of (f o g) = co-domain of f



$f \circ g : A \to C$ can be properly defined if range$(g) \subseteq$ domain(f)

Example.

$f : \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x - 4$

$g : \mathbb{Z} \to \mathbb{Z}$, where $g(x) = x^2 + 4$.

- $f \circ g : \mathbb{Z} \to \mathbb{Z}$ is properly defined, since the co-domain of g and the domain of f are the same ($\text{range}(g) = \{5, 8, 13, \dots\} \subseteq \mathbb{Z}$

$$f(g(x)) = f(x^2+4) = x^2$$

$$x \xrightarrow[g]{} x^2+4 \xrightarrow[f]{} (x^2+4)-4 = x^2$$

- $g \circ f : \mathbb{Z} \to \mathbb{Z}$ is properly defined, since the co-domain of f and the domain of g are the same ($\text{range}(f) = \mathbb{Z} \subseteq \mathbb{Z}$)

$$g(f(x)) = g(x-4) = (x-4)^2 + 4 = x^2 - 8x + 20$$

$$x \xrightarrow[f]{} x-4 \xrightarrow[g]{} (x-4)^2+4$$

Checklist

- Do you know what anti-symmetry is?
- Do you understand that symmetry and anti-symmetry are NOT opposites?
- Do you know when a relation is a partial order (and how to prove this)?

- Do you understand what a function is?
- Do you know the meaning of 'domain', 'co-domain' and 'range'?
- Do you understand how function composition works?