Course overview

- Logic (week 1)
- Proof techniques (weeks 1-2)
- Set theory (weeks 2-3)
- Relations today + next lecture
- Functions (week 4)
- Combinatorics (week 5)

Overview of today

- Relations: definition
- Reflexive relations
- Symmetric relations
- Transitive relations
- Equivalence relations

Book chapter 3, sections 3.1, 3.2, 3.3

Relations

A relation R describes the relationship between different elements of a given set A.

- Notation: $x \, R \, y \quad , x, y \in A$

- Example:

  $A = \{1, 2, 3\} \qquad xRy \text{ if } x \leq y$

  $1R2 \qquad 2 \not R 1 \qquad 3R3$

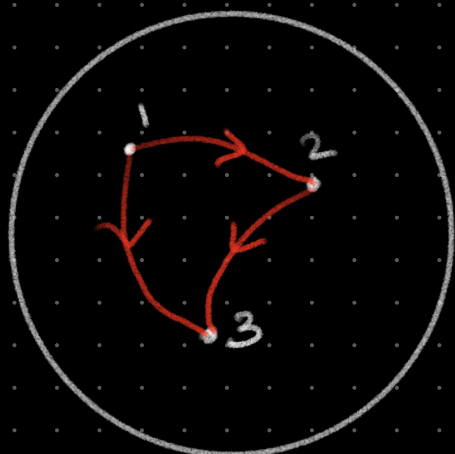
- Formal (abstract) definition: A relation R on a set A is a subset of the product set A x A.

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

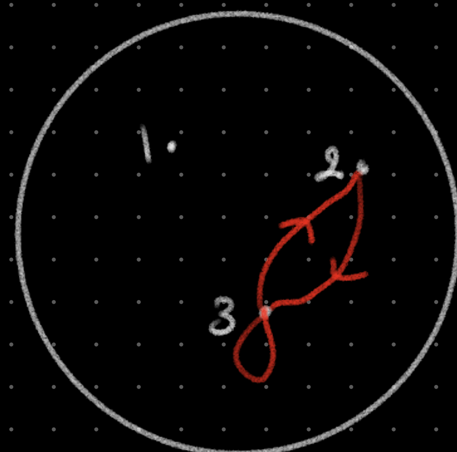
$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

Relation diagrams

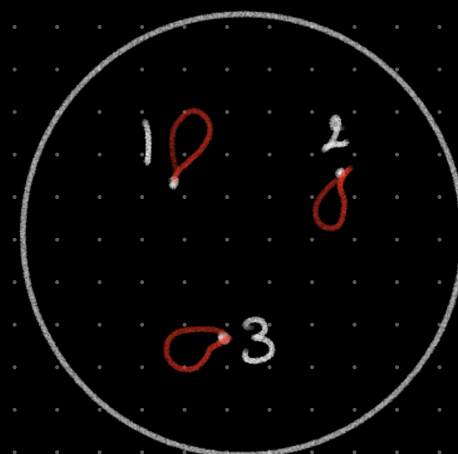
$A = \{1, 2, 3\}$



xRy if $x < y$



xRy if $x + y \geq 5$



xRy if $x = y$

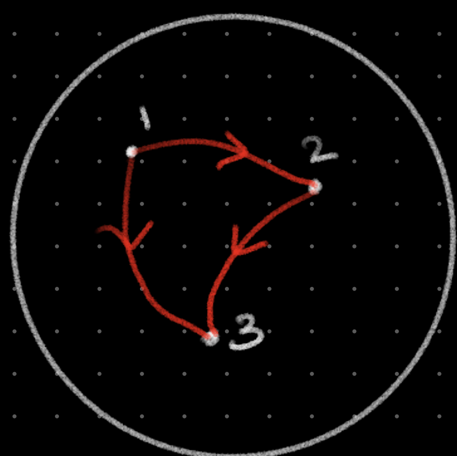
Reflexive relations

A relation R on a set A is **reflexive** if every element of A is related to itself

(every element has a loop)

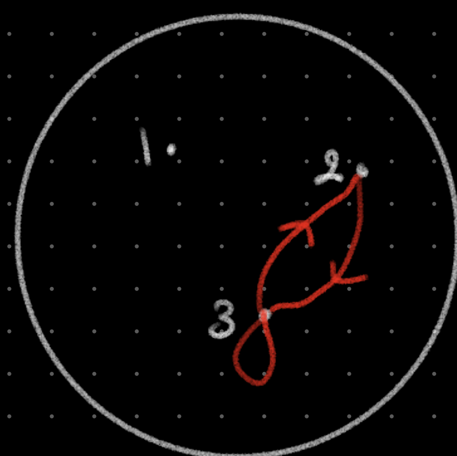
$\forall x \in A : xRx$

negation: $\exists x \in A : x \not R x$



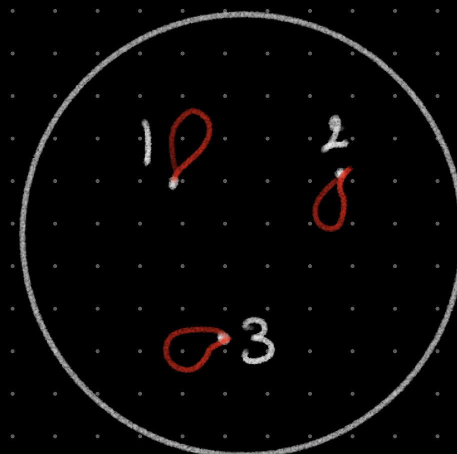
xRy if $x<y$

not reflexive



xRy if $x+y \geq 5$

not reflexive



xRy if $x=y$

reflexive

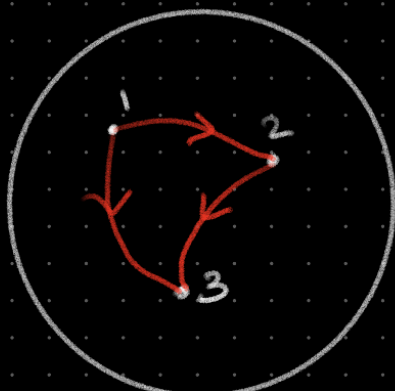
examples: $\leq$ (on $\mathbb{R}$), $\subseteq$ (on sets), ...

Symmetric relations

A relation R on a set A is symmetric if for all elements a,b in A:

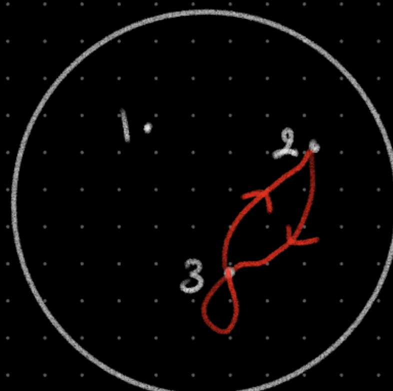
aRb ->bRa

$\forall a,b \in A : aRb \to bRa$ (every arrow goes in 2 directions)



xRy if $x<y$

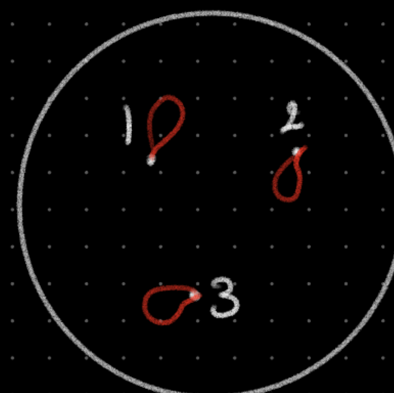
not symmetric



xRy if $x+y \geq 5$

symmetric

$$xRy \iff x+y \geq 5 \iff y+x \geq 5$$
$$yRx$$



xRy if $x=y$

symmetric

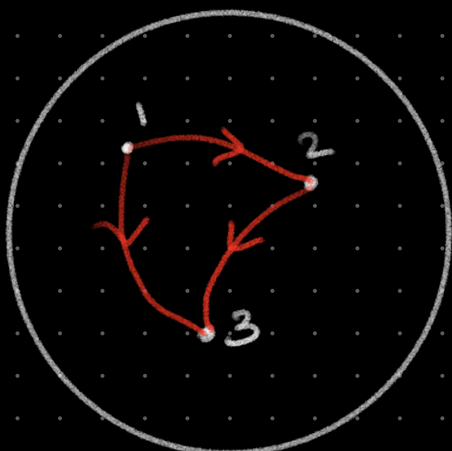
Transitive relations

A relation R on a set A is transitive if for all elements a,b,c in A:

(aRb ^ bRc) -> aRc

$$\forall a,b,c \in A : (aRb \wedge bRc) \longrightarrow aRc$$

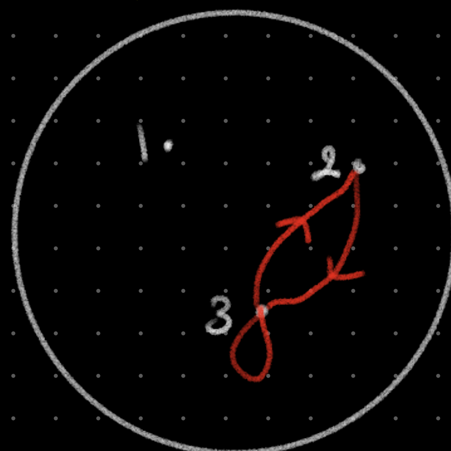
negation : $\exists a,b,c \in A : (aRb \wedge bRc) \wedge a \not R c$



xRy if $x<y$

transitive

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right, "then"'] & b \arrow[r] & c \end{tikzcd}$$

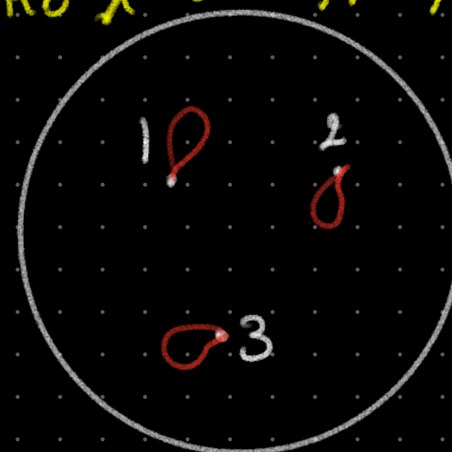


xRy if $x+y \geq 5$

not transitive

$2R3 \wedge 3R2$

but $2 \not R 2$



xRy if $x=y$

transitive

Equivalence relations

A relation R is called an equivalence relation if it is reflexive, symmetric and transitive.

Let $A = \mathbb{N}$

xRy iff $x-y$ is even

- reflexive? $\forall x \in A : xRx$

  Let $x \in \mathbb{N}$. $x-x=0$ is even, so xRx ✓

- symmetric? $\forall x,y \in A : xRy \rightarrow yRx$

  Assume $x,y \in \mathbb{N}$, xRy. Then $x-y$ is even
  then $y-x$ is even. So yRx ✓

- transitive? $\forall x,y,z \in A : (xRy \land yRz) \rightarrow xRz$

Assume $x, y, z \in \mathbb{N}$, $xRy \wedge yRz$

then $x-y$ is even and $y-z$ is even

then $(x-y)+(y-z)$ is even (sum of even numbers)

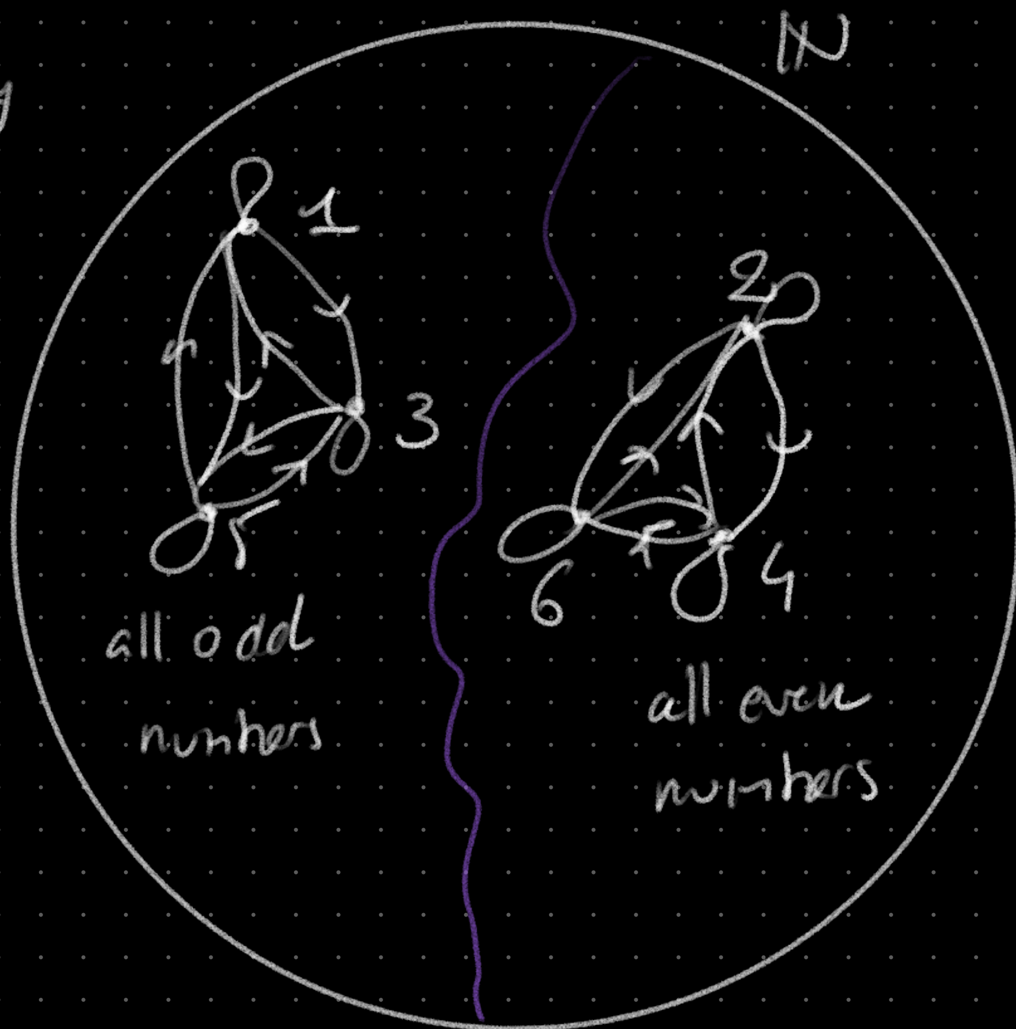
so $x-z$ is even. $\rightarrow xRz$ ✓

R is reflexive, transitive and symmetric

$\rightarrow$ it is an equivalence relation!!

Equivalence classes

xRy iff $x-y$ is even

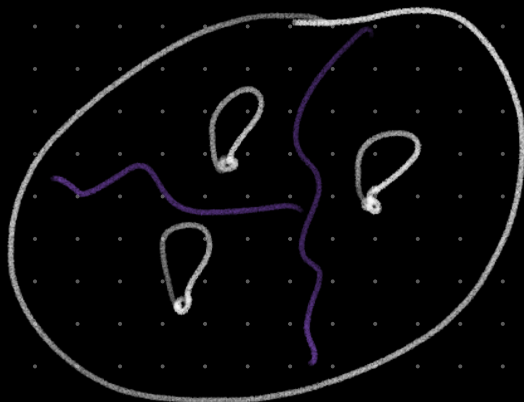


PARTITION of $\mathbb{N}$

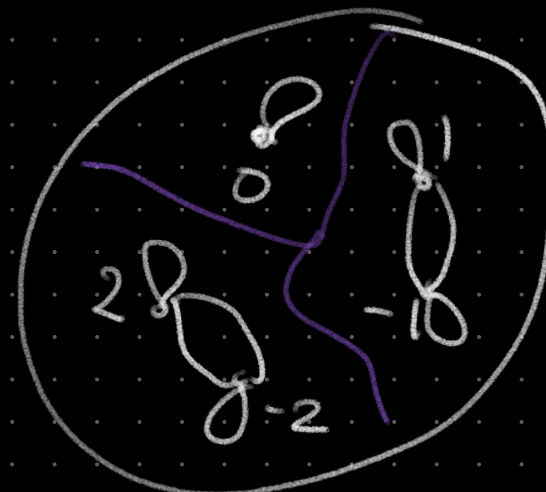
An equivalence relation on A induces a partition of A.

subsets = equivalence classes.

on $\mathbb{Z}$, xRy if $x^2 = y^2$



$x = y$



Checklist

- Do you understand what a relation is?
- Given a relation, are you able to draw a relation diagram?
- Do you know what symmetry, transitivity and reflexivity are?
- Can you prove (or disprove) that a relation has these properties?
- Do you know how to prove that a relation is an equivalence relation?
- Do you know how to identify the equivalence classes of an equivalence relation?

Example $\mathbb{R}\times\mathbb{R}$, $(x,y)R(a,b)$ if $x+y=a+b$

- reflexive: $\forall (x,y) \in \mathbb{R}\times\mathbb{R}$: $(x,y)R(x,y)$

  $(x,y)R(x,y)$ if $x+y = x+y$ true

- symmetric $\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}\times\mathbb{R}$,

  $(x_1,y_1)R(x_2,y_2) \to (x_2,y_2)R(x_1,y_1)$

Let

$(x_1,y_1)R(x_2,y_2)$. Then $x_1+y_1 = x_2+y_2$

then $x_2+y_2 = x_1+y_1$

so $(x_2,y_2)R(x_1,y_1)$ true.

- transitive $\forall (x_1,y_1),(x_2,y_2),(x_3,y_3) \in \mathbb{R}\times\mathbb{R}$

$$(x_1, y_1) R (x_2, y_2) \wedge (x_2, y_2) R (x_3, y_3) \longrightarrow$$
$$(x_1, y_1) R (x_3, y_3)$$

Let $(x_1, y_1) R (x_2, y_2) \quad \wedge \quad (x_2, y_2) R (x_3, y_3)$

then $x_1 + y_1 = x_2 + y_2 \quad \wedge \quad x_2 + y_2 = x_3 + y_3$

so $x_1 + y_1 = x_3 + y_3$

so $(x_1, y_1) R (x_3, y_3)$ true.

$(x,y) R (a,b)$ if $x+y = a+b$