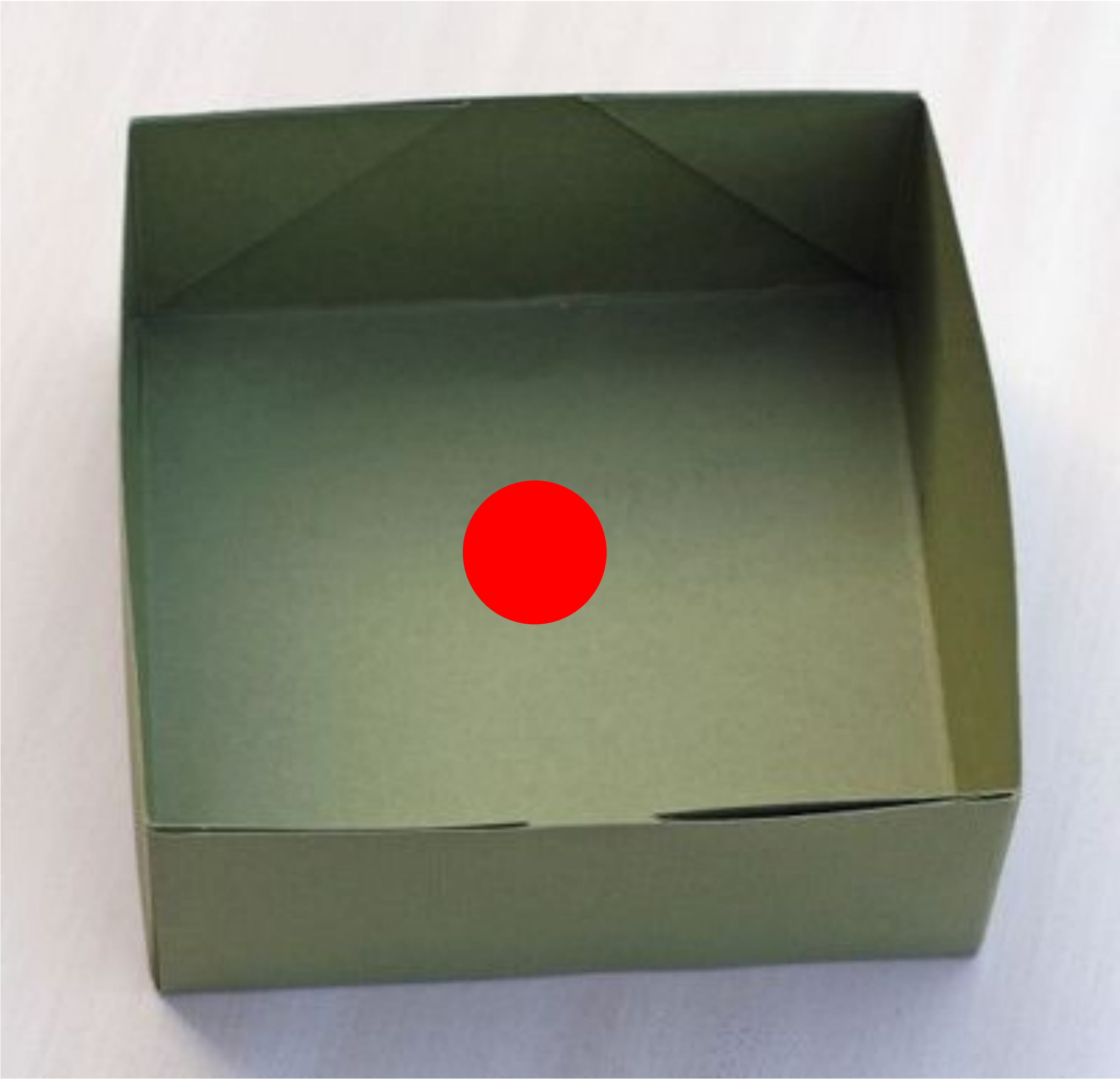
Lecture 1: recap

- Natural, whole and real numbers
- Divisibility and prime numbers
- Propositions
- Logical operations: and, or, not, implication, biconditional
- Truth tables

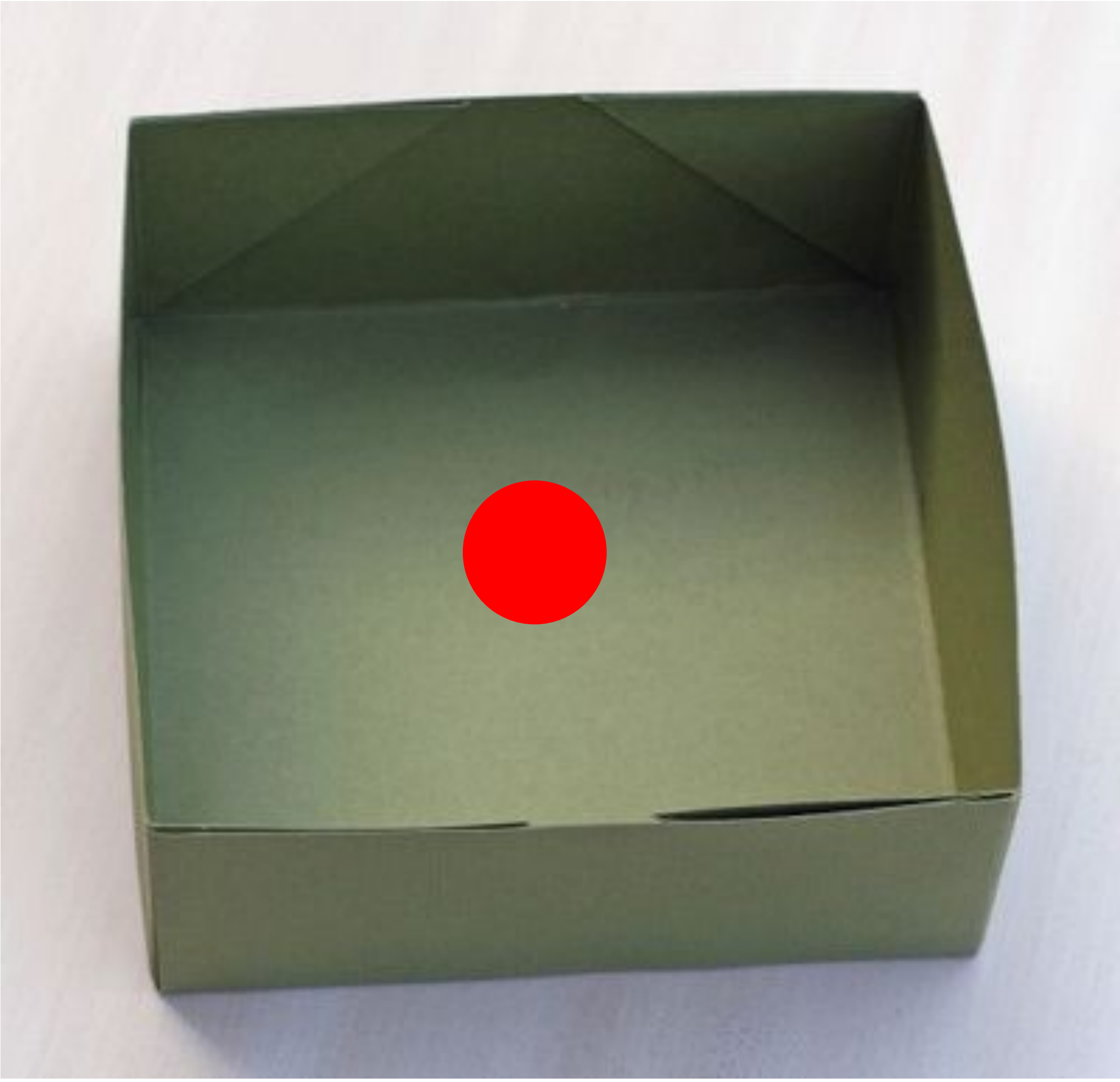
Lecture 2: Overview

- Quantifiers
- Proofs: direct proofs
- Counterexamples

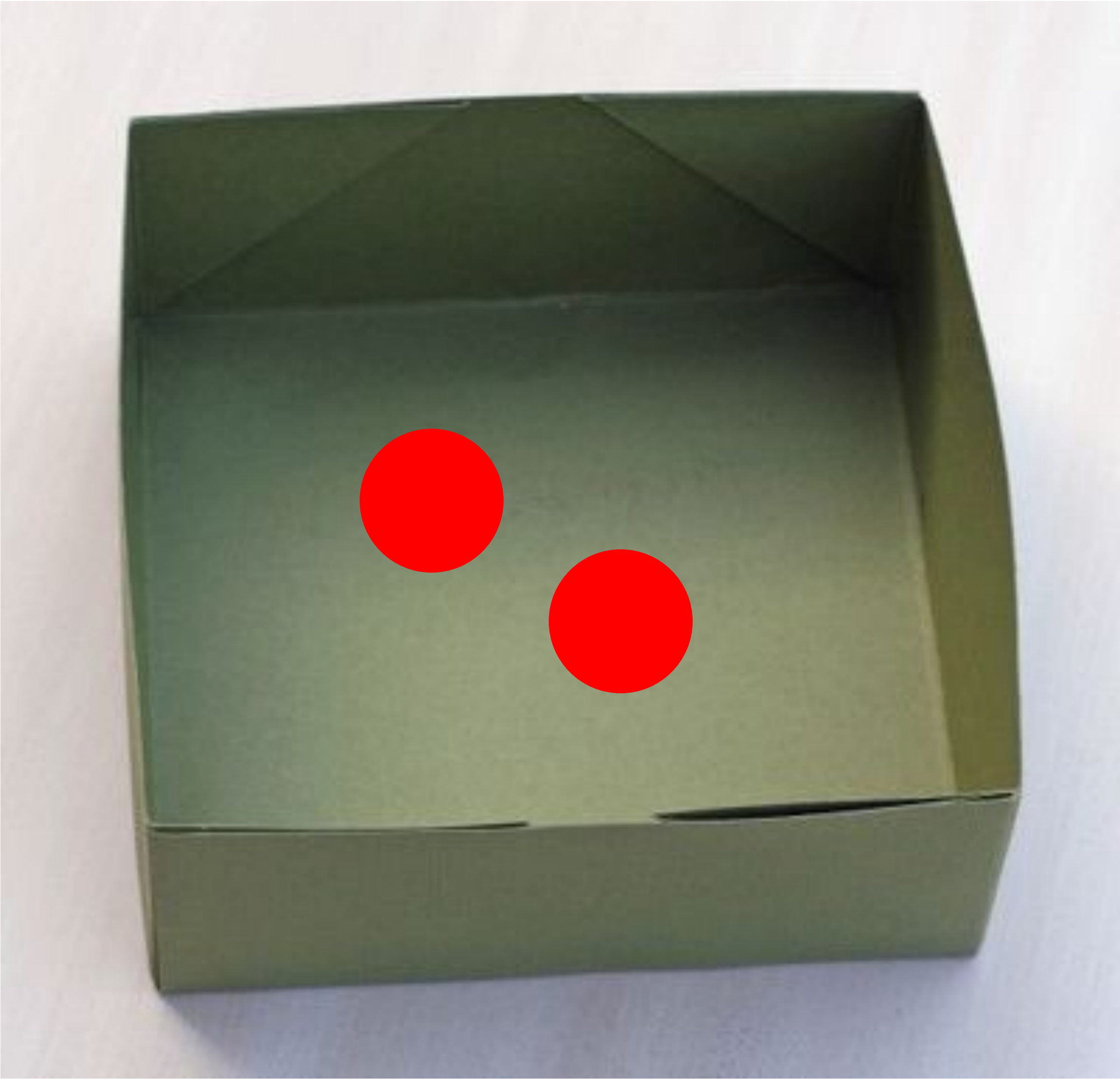
Book: Chapter 1, Sections 1.4, 1.5



There is a red ball in the box.



All the balls in the box are red.



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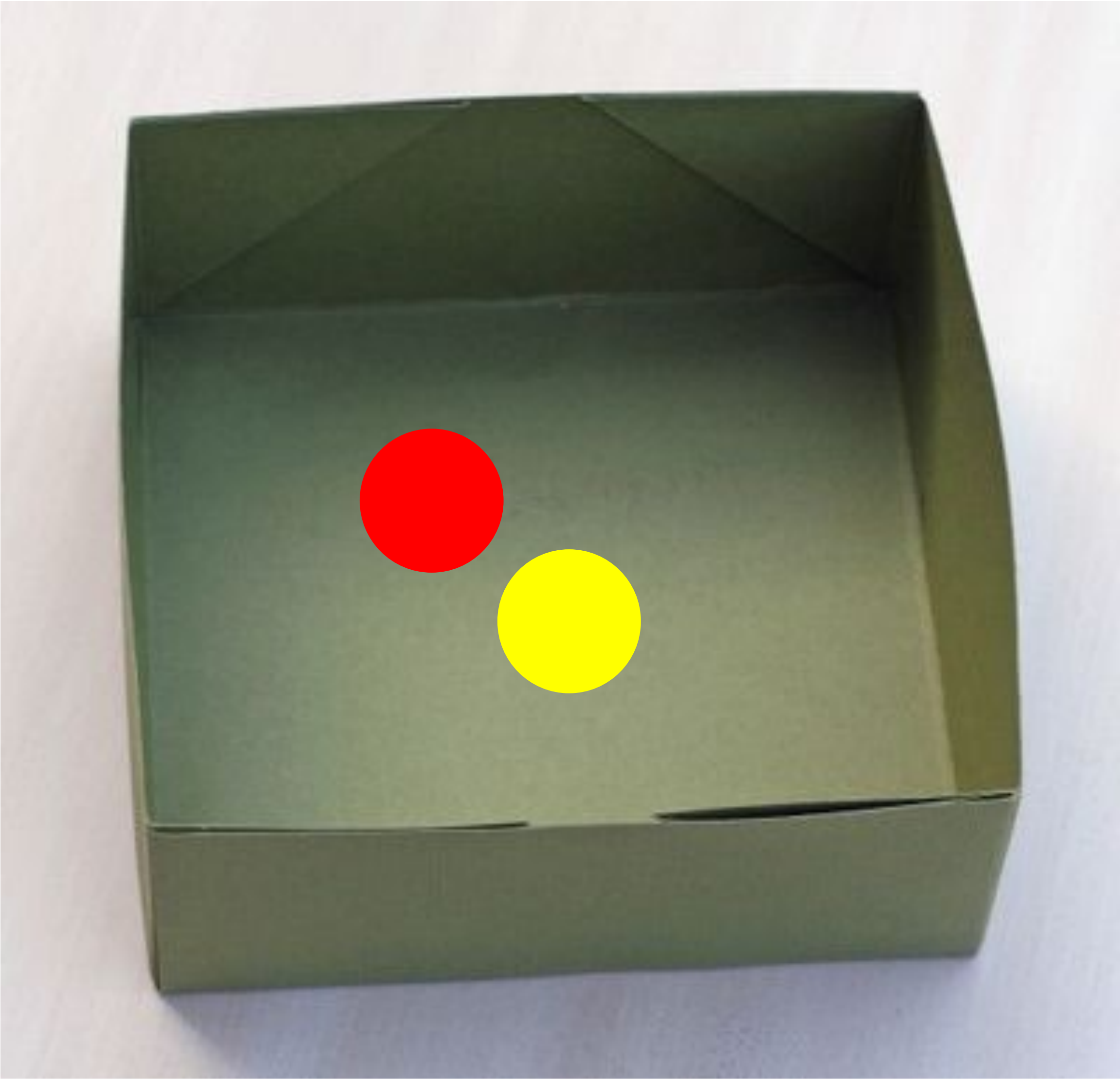
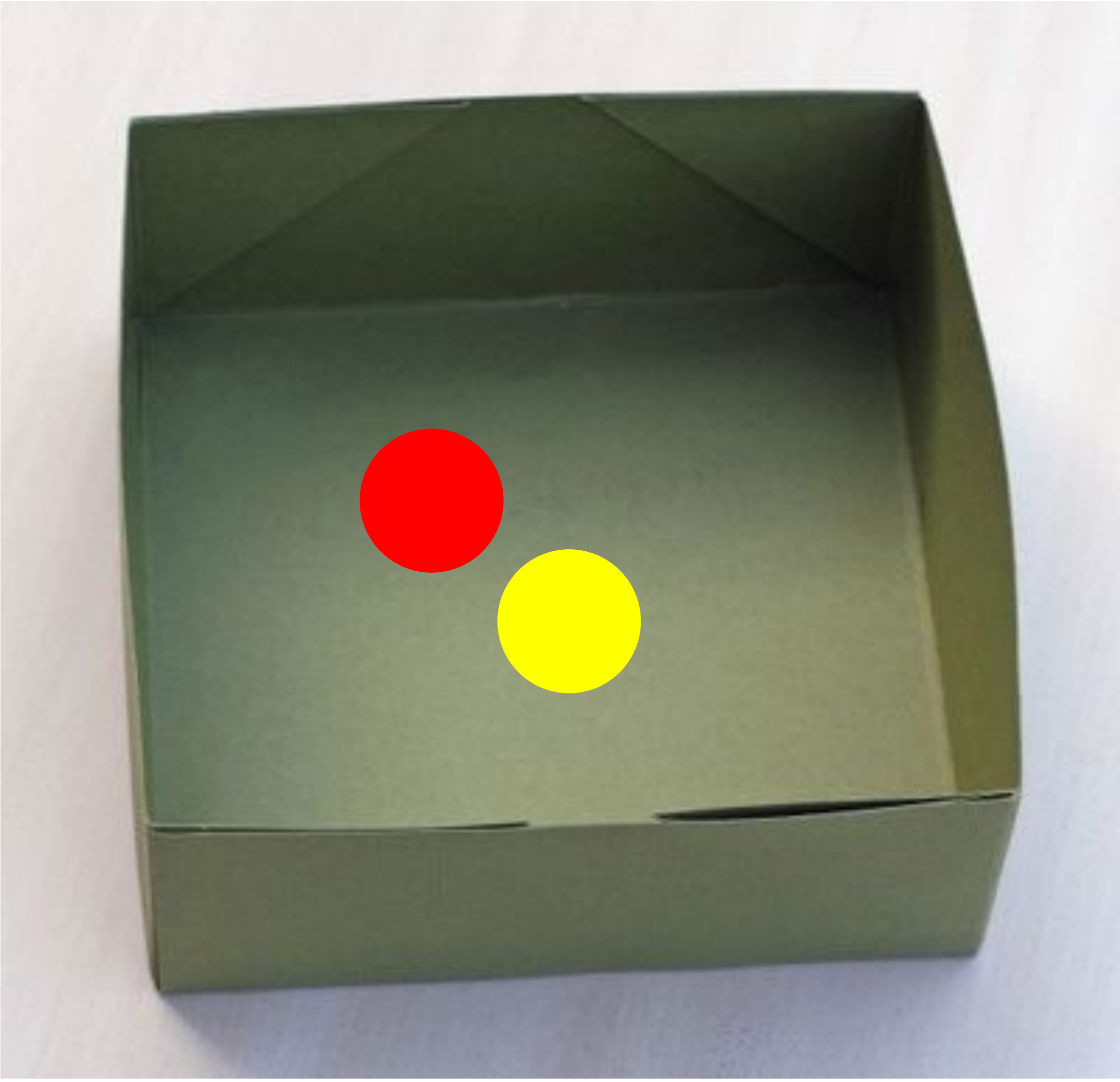
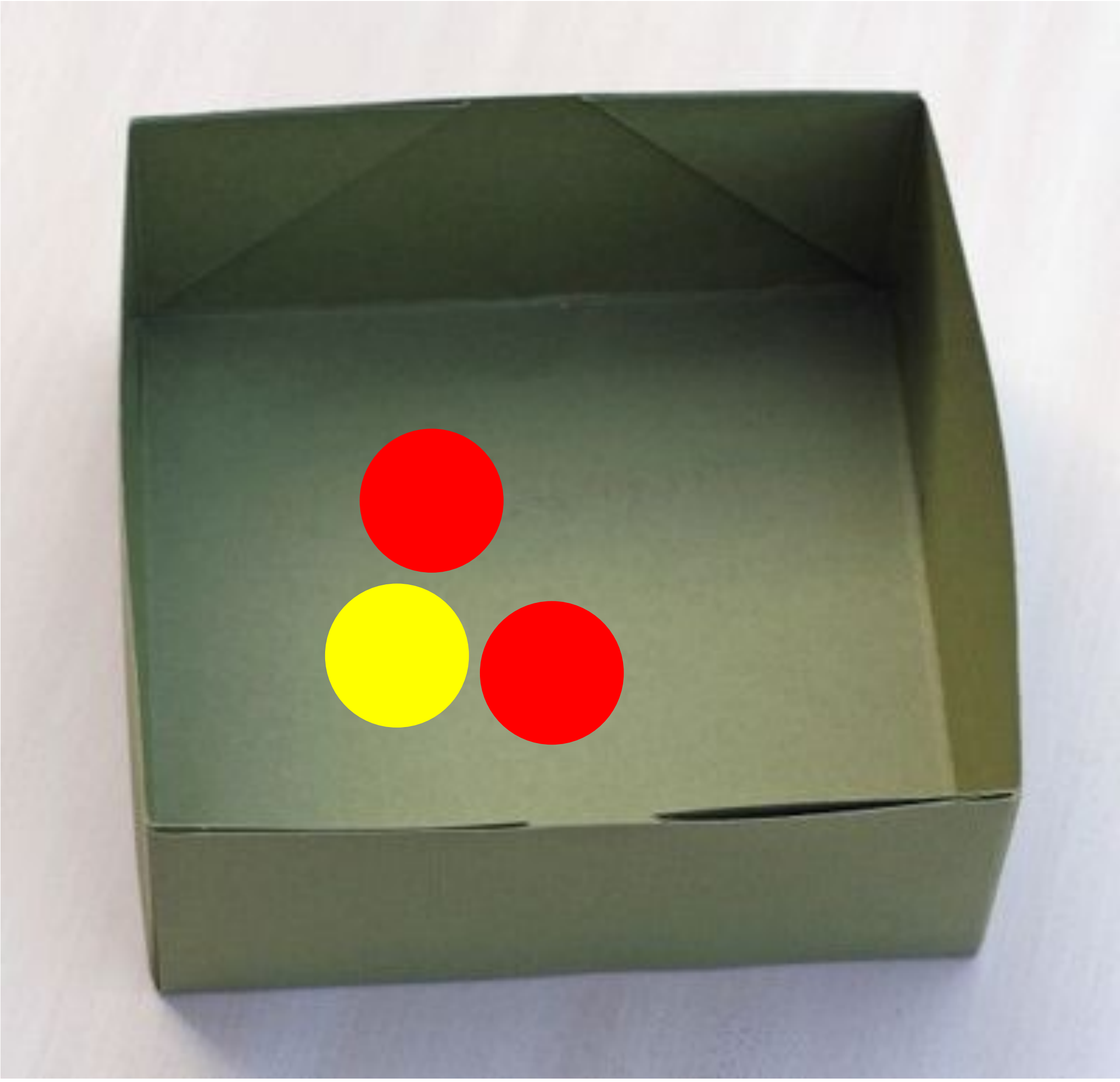


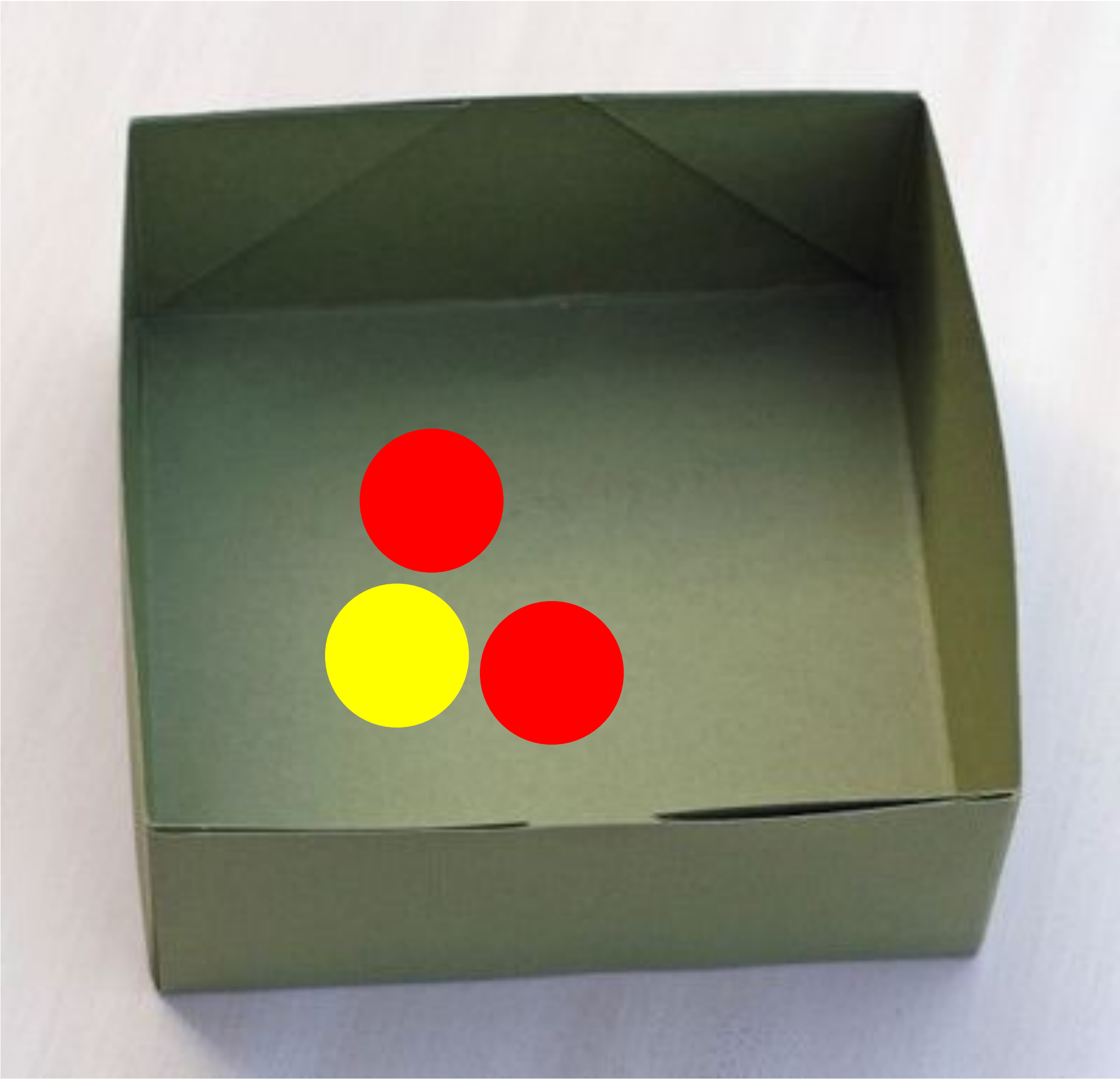
There is a red ball in the box.



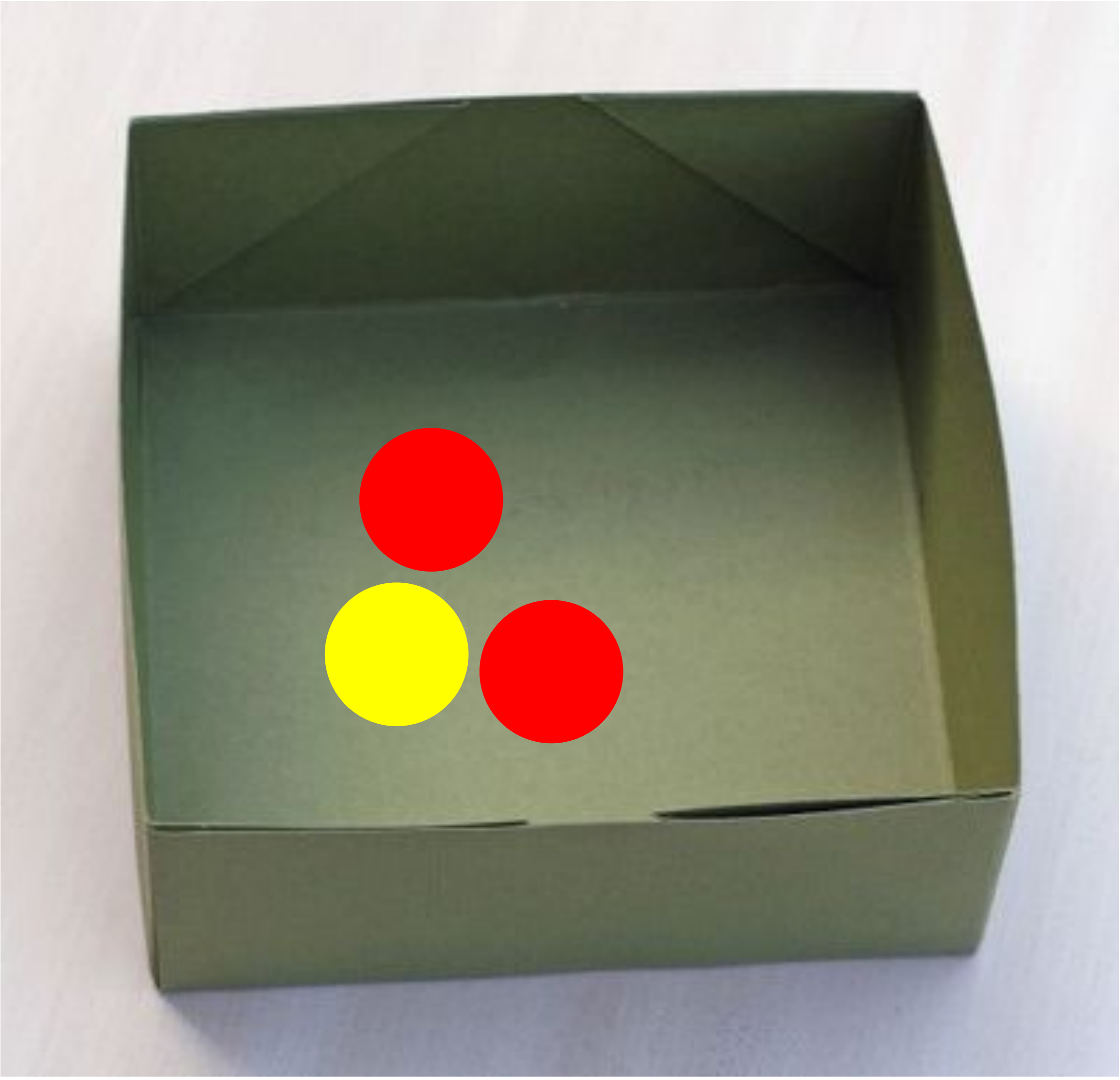
There is a red ball in the box and all the balls in the box have the same color.



Every ball in the box is red, or every ball in the box is yellow.



For each ball in the box, it is either red or yellow.

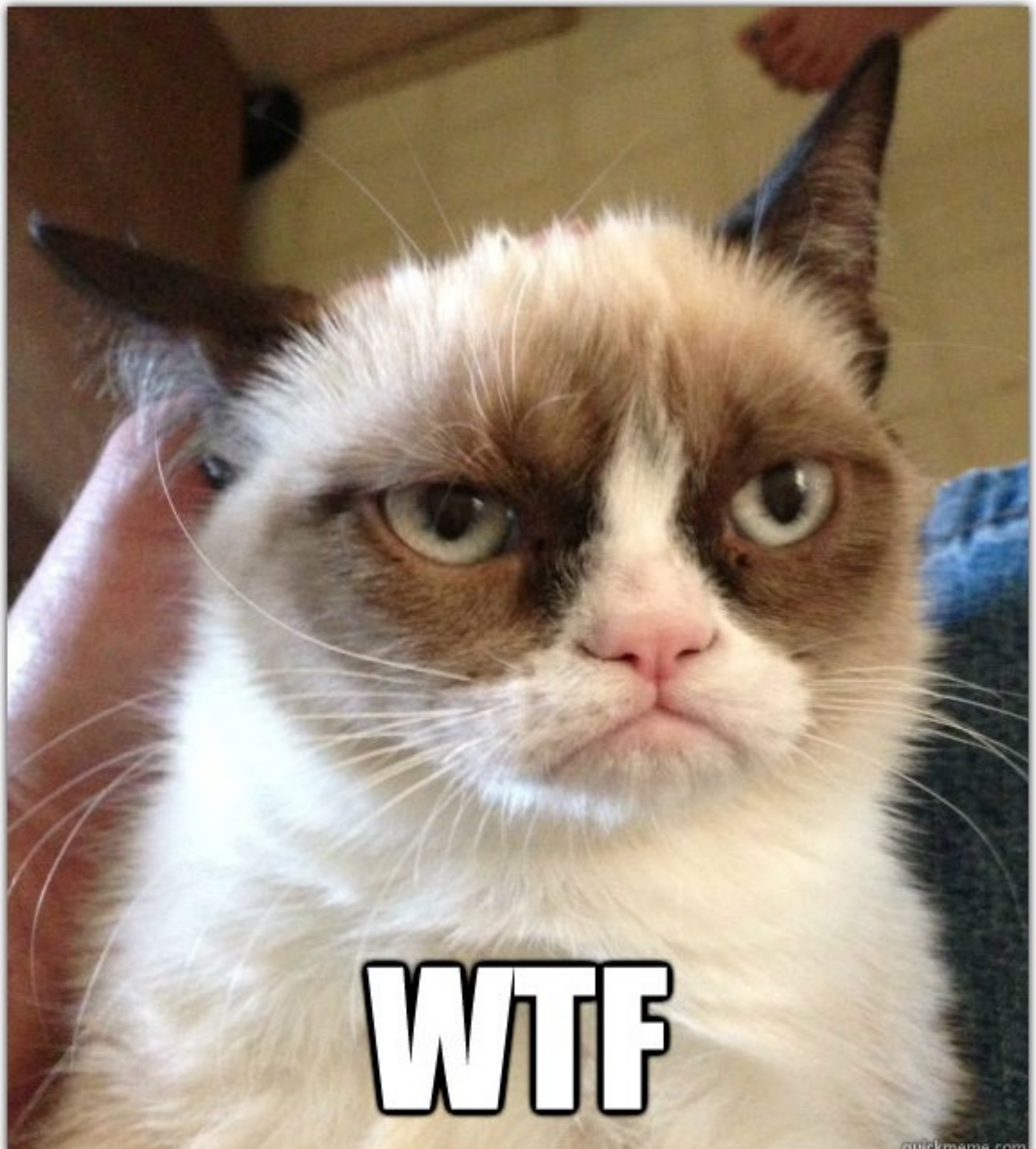


All the balls in the box are red.

True

All the balls in the box are red.

If a ball is in the box, it is red.


WTF
quickmeme.com

The box contains at least one ball, and every ball in the box is red.

Quantifiers

- for all $(\forall b \in box)(b \text{ is red})$ (universal)
  (all balls in the box are red)

- there exists $\exists b \in box,\ b \text{ is red}$ (existential)
  (there is a red ball in the box)

Examples:

All students at DACS are intelligent

$\forall x$ student at DACS, x is intelligent

x is student at DACS $\longrightarrow$ x is intelligent

$\exists x$ at DACS, x is intelligent

There is an intelligent student at DACS

Negation of quantifier statements

All balls in the box are red

There is a ball that is not red = not all balls are red

$\forall b \in box$: b is red

$\exists b \in box$: b is not red

Every country has a sea border

$\Leftrightarrow$ There is a country without sea border

($\forall$ countries)(country has sea border)

($\exists$ country) $\neg$ (country has sea border)

To negate, you flip the quantifiers and negate the body.
(The truth value also flips.)

Higher depth quantifiers

Every monkey climbs a tree

$(\forall m)(\exists t)(m \text{ climb } t)$

Negation: at least one monkey does not climb a tree

Every tree has a monkey in it

$(\forall t)(\exists m)(m \text{ climb } t)$

$(\exists t)(\forall m)(m \text{ climb } t)$

All monkeys climb the same tree

..)

The order of the quantifiers matters!

Every monkey climbs a tree

$(\forall m)(\exists t)(m \text{ climb } t)$

Negation: at least one monkey does not climb a tree

$(\exists m)(\forall t)\neg(m \text{ climbs } t)$

Note: if the quantifiers are the same, the order does not matter.

$\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}()$ is the same as $\exists y \in \mathbb{Z}\ \exists x \in \mathbb{Z}(\ )$

$\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}()$ is " $\forall y \in \mathbb{Z}\ \forall x \in \mathbb{Z}(..)$

Examples

$$\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} : x + y \geqslant 4$$

negation : $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} : x + y < 4$

$$\forall x \in \mathbb{N} \ \exists y \in \mathbb{N} \ \exists z \in \mathbb{N} : x^2 + y^2 = z^2$$

negation : $\exists x \in \mathbb{N} \ \forall y \in \mathbb{N} \ \forall z \in \mathbb{N} : x^2 + y^2 \neq z^2$

Proof techniques

- Direct proof
- Counterexample
- Contrapositive
- Contradiction
- Induction

today

tomorrow

next week

Direct proofs

- Move forward in logical steps from the assumption(s) to the conclusion.
- To prove "if p then q", you assume that p is true, and use this assumption to prove that q also holds
- To prove that a statement is false, you prove that the negation is true.

Example: $\forall n \in \mathbb{N} : n^2 \geq n$

- To prove a "for all" statement, you prove the statement for an arbitrary element.
- To prove a "there exists" statement, you only need to give an example.

Let $n \in \mathbb{N}$

then $n \geq 1$

then $n \cdot n \geq n \cdot 1$ (since $n > 0$)

$\Rightarrow n^2 \geq n$

Prove or disprove

$\forall x \in \mathbb{Z} \ \exists y \in \mathbb{Z} : 3x + y \leq 4$

Let $x \in \mathbb{Z}$

choose $y = -3x + 1 \in \mathbb{Z}$

and $3x + y = 3x + (-3x+1) = 1 \leq 4$

$\exists x \in \mathbb{Z} \ \forall y \in \mathbb{Z} : x + y = y + 1$

Choose $x = 1$. Then $1 \in \mathbb{Z}$ and $1 + y = y + 1$

$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z} : xy < y^2$

$\forall x \in \mathbb{Z} \ \exists y \in \mathbb{Z} : xy \geq y^2$

Let $x \in \mathbb{Z}$. Choose $y = x \in \mathbb{Z}$

then $xy = x^2 = y^2 \geq y^2$

Counterexamples

- To disprove a "for all" statement

Prove or disprove

- $\forall x \in \mathbb{N} : x^2 > 4$ counterexample: 1

  $1 \in \mathbb{N}, \quad 1^2 < 4$

- $\forall x \in \mathbb{R} : x^2 \geq x$ counterexample: 0.5

  $0.5 \in \mathbb{R}, \quad 0.5^2 = 0.25 < 0.5$

$\forall x \in \mathbb{H}, \exists y \in \mathbb{H} : x^y < y^x$

negation : $\exists x \in \mathbb{H}, \forall y \in \mathbb{H} : x^y \geqslant y^x$

for you to find out which one is true! The irst to show Otti a valid proof or worked out counter-example receives chocolate!

(offer valid until 15/09/2023)

Checklist

- Do you understand how quantifiers work?
- Can you negate quantifier statements?
- Do you understand why, if asked to prove something for all elements of a set, it is sufficient to start the proof by picking an arbitrary element?
- Do you understand why it is sufficient to disprove a "for all" statement to find a single counter-example?
- Are you comfortable proving statements of the form "if p, then q", by assuming p that is true, and showing that q follows?