Combinatorics (mathematics of counting)

- Rule of sum, rule of product
- Counting
  - With order, with repetition
  - With order, without repetition
  - Without order, without repetition
  - Without order, with repetition

Counting with/without order, with/without repetition

How many ways can I select 2 letters from the set {A, B, C}?

- order is important and repetition is allowed

  9 possibilities : AA AB AC BA BB BC CA CB CC

- order doesn't matter, and repetition is allowed

  AA AB AC BB BC CC 6 possibilities

  AB = BA, AC = CA, BC = CB

- order doesn't matter, and repetition is not allowed

  AB , AC , BC 3 possibilities

- order is important, repetition is not allowed

  AB AC BA BC CA CB 6 possibilities

How many different 5-letter, case-sensitive passwords can we generate?

- order matters (abcde is not the same as bacde)
- repetition is allowed (we can choose the same letter for multiple positions, aaAbb is a valid password)

→ we have
2×26 options for the first letter
2×26 second "
2×26 third
2×26 fourth
2×26 fifth

→ total: $52 \times 52 \times 52 \times 52 \times 52 = 52^5$ options
↳ rule of product.

If repetition is allowed, the number of ordered selections of k objects from a set of n objects is n^k

* here $n = 52$ (52 options for each letter)
$k = 5$ (you choose 5 letters)

A committee has 10 members. They want to select a chairperson, a treasurer and an administrator, and each member can do at most one of these jobs? How many different possibilities do they have?

- order matters (Otti as chair and Maricke as treasurer is not the same as Maricke as chair and Otti as treasurer)
- repetition is not allowed (one person cannot do two jobs)

→ 10 options for the chair
9 options treasurer (since 1 person is already chair)
8 options administrator

total: $10 \times 9 \times 8 = 720$ different committees.

↳ $\frac{10!}{7!}$

If repetition is not allowed, the number of ordered selections of k objects from a set of n objects is $n \times (n-1) \times \ldots \times (n-k+1)$.

↳ can be written as $\frac{n!}{(n-k)!}$

- notations: $(n)_k$, ${}^nP_k$ (permutation)
- here, $n = 10$ (10 persons to choose from), $k = 3$ (3 jobs)

An exam has 7 questions. A students has to answer 4. How many ways can this happen?

- repetition is not allowed (you cannot solve Q1 4 times)
- order is not important (answering Q1, Q3, Q5, Q7 is the same as Q1, Q7, Q5, Q3)

→ if order mattered, we had $7 \times 6 \times 5 \times 4$ ways

↳ we need to correct for counting the same set of questions multiple times.

$$\left.\begin{array}{llll} Q1 & Q3 & Q5 & Q7 \\ Q1 & Q7 & Q5 & Q3 \\ Q1 & Q7 & Q3 & Q5 \\ \vdots & & & \vdots \end{array}\right\} \text{we have } 4! \text{ different orders}$$

↳ so we correct overcounting by dividing by 4!

answer: $\dfrac{7 \times 6 \times 5 \times 4}{4!} = \dfrac{7 \times \cancel{6} \times 5 \times \cancel{4}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = 35$ ways

If repetition is not allowed and order does not matter, the number of selections of k objects from a set of n objects is $\dfrac{n!}{k!\,(n-k)!}$

here: $n = 7$, $k = 4$

notations: $\binom{n}{k}$, ${}^{n}C_{k}$ (combination)

We have 3 identical dice. If we throw them at the same time, how many different patterns are possible?

- order does not matter (4 6 4 is the same as 4 4 6)
- repetition is allowed (you can roll more than one 6)

→ all possible outcomes can be summarized in a table

1	2	3	4	5	6
x		x	x		
	xx			x	

→ 1 3 4   x | | x | x | |
2 2 5 | x x | | | x |

every outcome corresponds (1 on 1) to a pattern of 5 | and 3 x

↳ the number of outcomes = the numbers of patterns consisting of 5 bars, 3 crosses

↳ 8 positions, you need to select 3 positions to put the x (repetition not allowed, order does not matter)

$$\rightarrow \frac{8!}{5!\,3!} = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{6}} = 56$$

When repetition is allowed and order does not matter, the number of ways to select k objects from a set of n objects is given by $\dfrac{(n-1+k)!}{k!\,(n-1)!}$

- $n-1$ bars (column dividers) for n options
- k crosses

$n-1+k$ objects in the strin

How many different numbers between 0 and 1000 have exactly one 3?

- 3 can be the 1^{st}, 2^{nd} or 3^{rd} digit.

81 numbers — 81 options — 81

$\longrightarrow 3 \times 81 = 243$ numbers.

How many ways can 6 people sit at a round table?
If two people insist on sitting together, how many options are left?

Diy

How many ways can we distribute 10 chocolates over 7 people? 8008
What if each person gets at least one chocolate? 84
What if each person gets at most two chocolates? 161

CASE I

1) Repetition allowed (a person can get more then 1 chocolate)
2) order not important (giving the 1st chocolate to A and the 2^{nd} to B, or vice versa, is the same situation)

↳ "textbook" unordered selection with repetition, $n = 7$ (7 persons you can choose to make happy with chocolate), $k = 10$ (10 time you pick a person) (make a choice)

$$\frac{(n+k-1)!}{k!\,(n-1)!} = \frac{16!}{10!\,6!} = 8008$$

A	B	C	D	E	F	G
x	x x	x	xx x	x x		x

an example of chocolate distribution.

CASE II

- we have no options for the first 7 chocolates (one for each)
- we can distribute 3 chocolates over 7 persons, order does not matter, repetition allowed

↳ unordered selection with repetition, $n = 7$, $k = 3$

$$\frac{(n+k-1)!}{k!\,(n-1)!} = \frac{9!}{3!\,6!} = \frac{9 \cdot 8 \cdot 7}{6} = 84$$

CASE III (you need insight and creativity)

1. • 5 people receive 2 chocolates, 2 receive nothing
2. • 4 people receive 1 chocolate, 3 receive 2 chocolates
3. • 4 people receive 2 chocolates, 2 receive 1 chocolate, 1 receives nothing

↳ these are the only options

→ case 1: select 2 poor persons from a group of 7 (no repetition, no order) → $\frac{7 \times 6}{2}$ ways $= 21$

→ case 2: select 3 fortunate persons from 7 (no repetition, no order) → $\frac{7 \times \cancel{6} \times 5}{\cancel{6}} = 35$ ways

→ case 3: 7 options to choose the person without chocolate

$\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ ways to choose 2 persons from the 6 that are left, who receive 1 choc.

↳ 7×15 ways $= 105$ ways

total (rule of sum): $105 + 35 + 21 = 161$ possibilities

5p **5c** We have 3 colours of paint available, red, blue and green. There is an unlimited amount of each available, but we are only allowed to buy whole litres of paint. We want to mix these paints to obtain a total volume of 8 litres. The colour of the paint we obtain is uniquely determined by the amount of red, blue and green in it. (So, “Three litres red, two litres blue, three litres green" will generate one colour, and “Two litres red, five litres blue, one litre green" will generate another colour.) How many different colours can we obtain?

- (a) 45
- (b) 56
- (c) 120
- (d) 336
- (e) 512
- (f) 6561
- (g) None of the above.

• repetition allowed

• order does not matter

$n = 3$

$k = 8$

$$\frac{(n+k-1)!}{(n-1)!\,k!} = \frac{10!}{8!\,2!} = \frac{10 \cdot 9}{2} = 45$$

R	G	B
x x x x	x x x x	

x x x x | x x x x |

Enjoy your project week!